LUND UNIVERSITY

FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 9: Control design for nonlinear systems

YIANNIS KARAYIANNIDIS, ASSOCIATE PROFESSOR
AUTOMATIC CONTROL, FACULTY OF ENGINEERING.

www.yiannis.info
yiannis@control.lth.se

Outline

- Lyapunov-based control design
- Exact feedback linearization

Lyapunov-based design

$\dot{x} = f(x, u)$

Steps of Lyapunov-based design:

1. Select a positive definite $V(x)$.
2. Calculate $\dot{V}(x) = \frac{\partial V}{\partial x} f(x, u)$.
3. Find a (possibly) nonlinear feedback control law that makes $\dot{V}$ negative.

- $\dot{V} \leq 0 \longrightarrow x = 0$ may be asymptotically stable (check LaSalle)
- $\dot{V} < 0$ for all $x \neq 0 \longrightarrow x = 0$ asymptotically stable
- $\dot{V} \leq -\lambda V \longrightarrow x = 0$ exponentially stable if addtionally $V \geq c\|x\|^2$

Comments:

- Selection of $V(x)$
- Depends on the system dynamics $\dot{x} = f(x, u)$

Example 3 (Lyapunov-based design)

Consider the system

$V = \frac{1}{2}(x_1 - x_1^*)^2 + \frac{1}{2}(x_2 - x_2^*)^2$

$$\dot{x}_1 = x_2^3$$
$$\dot{x}_2 = u$$

$\dot{x} = f(x, u) = \begin{bmatrix} x_2^3 \\ u \end{bmatrix}$

For (*)
$\begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = -x_1 - k x_2 \end{cases}$

Find a globally asymptotically stabilizing control law $u = u(x)$. $x^* = 0$

1. $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ p.d $x^* = 0$

2. $\dot{V} = \frac{\partial V}{\partial x} \cdot \dot{x} = x_1 x_2^3 + x_2 u$

$u = -x_1 x_2^2 - \cancel{x_1} x_2^2$

$\dot{V} = -x_1^2 x_2^2$

$\begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = \cancel{-x_1^2 x_2^2} - x_1 x_2^2 - x_1 x_2^2 \end{cases}$

correction

$E = \{\dot{V} = 0, x_1^2 x_2^2 = 0\}$

Note the equilibrium is $(x_1, 0)$

$u = -x_1 x_2^2 - k x_2$

$\dot{V} = -k x_2^2$

$\begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = -k x_2 - x_1 x_2^2 \end{cases}$ (*)

$E = \{x \in \mathbb{R}^2, x_2 = 0\}$ (*) $\Rightarrow M = \{x_1 \in \mathbb{R}, x_2 = 0\}$

choose another

$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^4$

$\dot{V} = x_1 \dot{x}_1 + x_2^3 \dot{x}_2$

$= \underbrace{x_1 x_2^3} + x_2^3 u.$

$u = -x_1 - k x_2$ (*)

$\Rightarrow \boxed{\dot{V} = -k x_2^4}$

$E = \{x \in \mathbb{R}^2, x_2 = 0\}$

$M = \{(0,0)\}$

LUND UNIVERSITY

Energy shaping (nonlinear spring)

$x_1 = x \quad x_2 = \dot{x}$

$z(x) = kx^3 \qquad z(x) \in [0, \infty]$

Friction-less system: $m\ddot{x} = -z(x) + u$

if $z(x) = kx$

$m = 1$

Total energy : $E = \frac{1}{2}\dot{x}^2 + \underbrace{\int_0^x z(\sigma)\sigma}$ → potential energy

Energy derivative along trajectories: $\dot{E} = \dot{x}u \quad (*)$

Control the energy to some desired level E_d

New Lyapunov function: $V = \frac{1}{2}(E - E_d)^2$

p.d. for $\dot{E} = \dot{x}u.$

$\dot{V} = \dot{E}(E - E_d)$

$= \dot{x}(E - E_d) \cdot u$

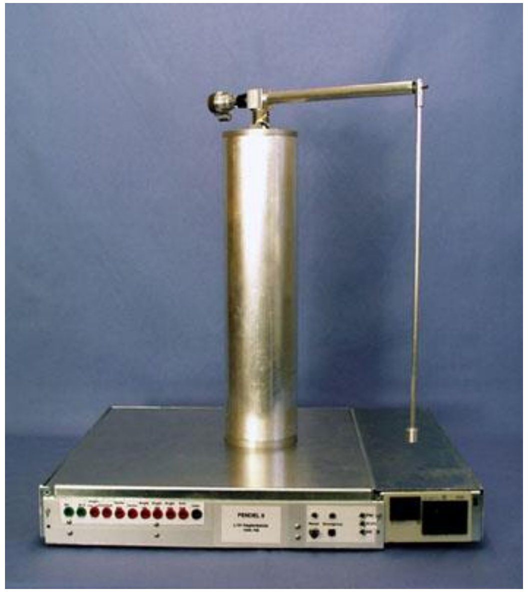
$u = -(E - E_d)$

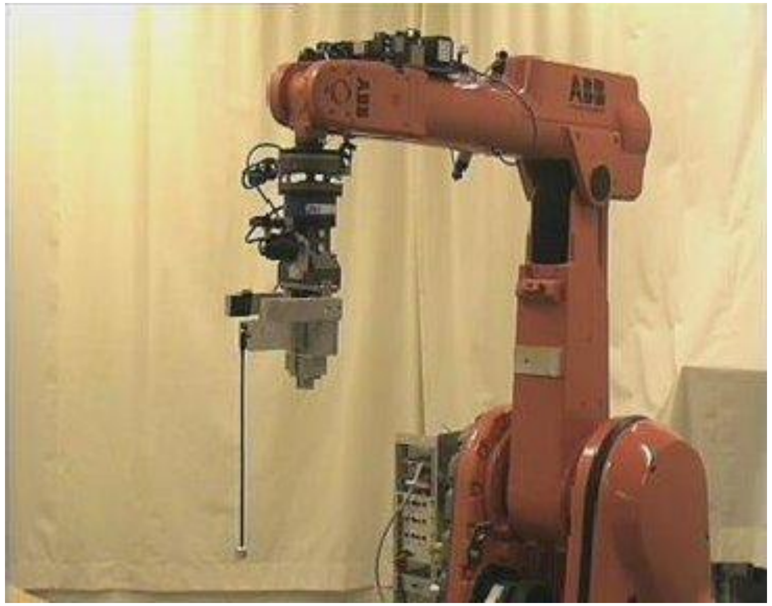
$\dot{V} = -\dot{x}(E - E_d)^2$

$u = -\dot{x}(E - E_d)$

$\dot{V} = -\dot{x}^2(E - E_d)^2.$

Energy shaping (swing-up control)





Rough outline of method to get the pendulum to the upright position

- Find expression for total energy E of the pendulum (potential energy + kinetic energy)
- Let E_n be energy in upright position.
- Look at deviation $V = \frac{1}{2}(E - E_n)^2 \geq 0$
- Find ”swing strategy” of control torque u such that $\dot{V} \leq 0$

LUND UNIVERSITY

Exact feedback linearization

- Find state feedback $u = u(x, v)$ so that the nonlinear affine in the control system

$$\dot{x} = f(x) + g(x)u$$

  turns into the linear system

$$\boxed{\dot{x} = Ax + Bv}$$

  and then apply linear control design method.

- Not all system can be exactly linearized. There are systems that their state needs to transformed to become linearizable:
  - first find $\boxed{z = T(x)}$ such that $\dot{z} = F(z) + G(z)u$
  - then find u that $\dot{z} = Az + Bv$
  - Design v as linear feedback controller with feedback of z.

Exact feedback linearization

- Find state feedback $u = u(x, v)$ so that the nonlinear affine in the control system

$$\dot{x} = f(x) + g(x)u$$

turns into the linear system

$$\dot{x} = Ax + Bv$$

and then apply linear control design method.

- Ax can be:
  - The linear part of the nonlinear system, e.g. $f(x) = Ax + \bar{f}(x)$
  - The linearized nonlinear system, e.g $f(x) = Ax + \bar{f}(x)$ where $\bar{f}(x) - \frac{\partial f}{\partial x}x$.
  - A desired linear dynamics specification.

LUND UNIVERSITY

Exact feedback linearization

- Relative degree 1: For $g(x)$ square and invertible

$$\boxed{\dot{x} = f(x) + g(x)u}$$

$x \in \mathbb{R}^4$ $\quad u \in \mathbb{R}^4$

$$u = g^{-1}(x)\left[-f(x) + v\right]$$

First order integrator

$$\boxed{\dot{x} = v}$$

- Relative degree n

$$\xi^{(n)} = f(\xi, \dot{\xi}, \cdots, \xi^{(n-1)}) + g((\xi, \dot{\xi}, \cdots, \xi^{(n-1)}))u$$

$$x = \left[\xi, \dot{\xi}, \cdots, \xi^{(n-1)}\right]^T$$

$$u = g^{-1}(x)\left[-f(x) + v\right]$$

$$\dot{x} = \begin{bmatrix} \mathbf{0}_{n-1} & \boldsymbol{I}_{n-1} \\ 0 & \mathbf{0}_{n-1}^T \end{bmatrix} x + \begin{bmatrix} \mathbf{0}_{n-1} \\ f(x) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{n-1} \\ g(x) \end{bmatrix} u$$

n-th order integrator

$$\xi^{(n)} = v$$

$u = h(x_2) + z(x_1) + v$

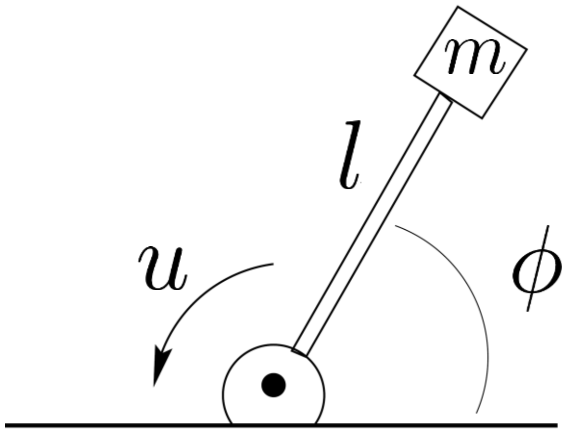
$\dot{x} = \begin{bmatrix} 0 \\ v \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} \mathbf{0}_{n-1} & \boldsymbol{I}_{n-1} \\ 0 & \mathbf{0}_{n-1}^T \end{bmatrix} x + \begin{bmatrix} \mathbf{0}_{n-1} \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -h(x_2) - z(x_1) + u \end{cases} \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -h(x_2) - z(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

LUND UNIVERSITY

Exact feedback linearization



An inverted pendulum controlled by a motor torque u at the joint:

$$\ddot{\phi}(t) = \frac{g}{l}\sin(\phi(t)) + \frac{1}{ml^2}u,$$

$x_1 = \phi$
$x_2 = \dot{\phi}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{g}{l}\sin(x_1) - \frac{1}{ml^2}u \end{cases}$$

Control structure for exact feedback linearization: $u = ml^2\left[\frac{g}{l}\sin(x_1) - v\right]$

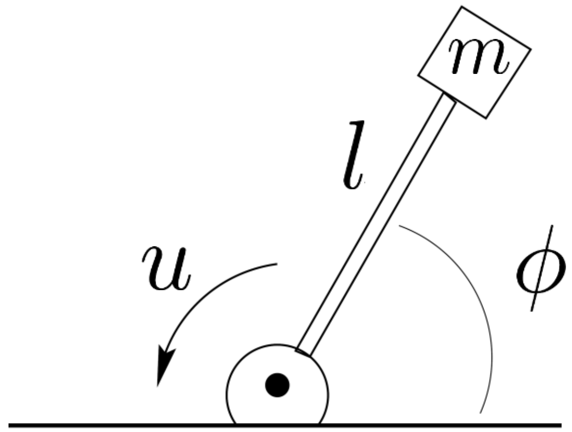
$\dot{x}_1 = x_2$

$\dot{x}_2 = \frac{g}{l}\cancel{\sin(x_1)} - \left[\frac{g}{l}\cancel{\sin(x_1)} - v\right] = v$

Then v is chosen as $v = -k_1(\phi - \delta) - k_2\dot{\phi}$

Exact feedback linearization and control-design based on linearization

An inverted pendulum controlled by a motor torque u at the joint:



$$\ddot{\phi}(t) = \frac{g}{l}\sin(\phi(t)) + \frac{1}{ml^2}u,$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g}{l}\sin(x_1) + \frac{1}{ml^2}u \end{aligned}$$

Control structure for exact feedback linearization:

$$u = ml^2\left[-\frac{g}{l}\sin(x_1) + v\right]$$

Closed loop system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = v \end{cases}$$

Linear

Control design based on linearization:

$$u = ml^2\left[-\frac{g}{l}\sin\delta + v\right]$$

δ is where we want to stabilize the system.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = v + \frac{g}{l}(\sin(x_1) - \sin(\delta)) \end{cases}$$

Non-linear system

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos\phi & 0 \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x}(\delta, 0) = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos\delta & 0 \end{bmatrix}$$

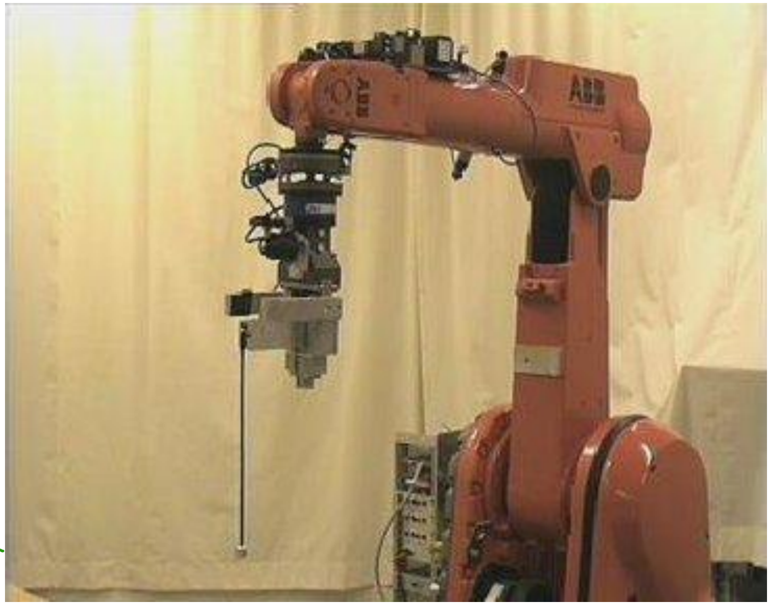
Multi–joint robot control with exact feedback linearization

Dynamic model of the robotic arm:

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = u, \qquad \theta \in R^n$$

Called *fully* actuated if n indep. actuators,

$M(\theta)$	$n \times n$ inertia matrix, $M = M^T > 0$
$C(\dot{\theta},\theta)\dot{\theta}$	$n \times 1$ vector of centrifugal and Coriolis forces
$G(\theta)$	$n \times 1$ vector of gravitation terms



Design a controller so that $\theta \longrightarrow \theta_r$.

Inverse dynamics approach.

$$v = -k_p(x_1 - x_{1r}) - k_v x_2$$
$$= -k_p(\theta - \theta_r) - k_v \dot{\theta}$$

$$K_p = \begin{bmatrix} k_{p1} & & \\ & \ddots & \\ & & k_{pn} \end{bmatrix} \quad K_v$$

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = u.$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = M(x_1)^{-1}\left(-C(x_1,x_2)x_2 - G(x_1) + u\right) \end{cases}$$

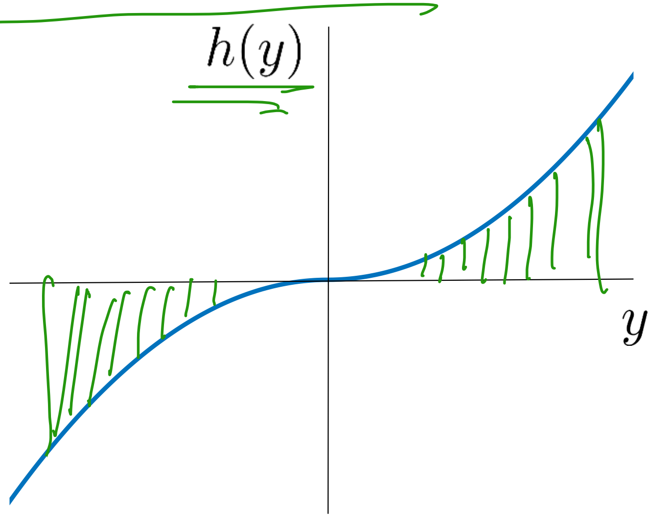
$$u = M(x_1)\left(+C(x_1,x_2)x_2 + G(x_1) + v\right)$$

$$\dot{x}_2 = \cancel{M(x_1)^{-1}}\,\cancel{M(x_1)}\,v$$
$$= v$$

LUND UNIVERSITY

Should I cancel or not?

- Good nonlinearities – passive



$[0, \infty]$

$$\ddot{x} = -\boxed{h(\dot{x})} - \boxed{z(x)} + u$$

nonlinear damping nonlinear spring

$$\dot{x}h(\dot{x}) \geq 0$$

Passive: energy absorbed by the damper is positive

$$\dot{x}z(x) = \frac{dP}{dt}$$

Passive: energy stored in the spring is positive

$$P = \int_0^x z(\sigma)\sigma$$

Total energy as Lyapunov function: $\boxed{V = \frac{1}{2}\dot{x}^2 + \int_0^x z(\sigma)d\sigma}$

Energy derivative along trajectories: $\dot{V} = -\dot{x}h(\dot{x}) + \dot{x}u$

$u = -\varphi_D(\dot{x})$ $\qquad \varphi_D(\dot{x}) \in (0 \quad \infty)$

$u = -k\dot{x}$

$\dot{V} = -\dot{x}h(\dot{x}) - \dot{x}\varphi_D(\dot{x})$

$V_1 = \frac{1}{2}\dot{x}^2 + \int_0^x z(\sigma)\,d\sigma + \int_0^x \phi_P(\sigma)\,d\sigma$

What if I choose: $u = -\varphi_P(x)$

e.g. $u = -kx$

$\phi_P(x) \in [0, \infty]$

Should I cancel or not?

Total energy as Lyapunov function: $V = \frac{1}{2}\dot{x}^2 + \int_0^x z(\sigma)\sigma$

Energy derivative along trajectories: $\dot{V} = -\dot{x}h(\dot{x}) + \dot{x}u$

$u = -\varphi_D(\dot{x})$ $\quad \varphi_D(\dot{x}) \in (0 \quad \infty)$

$\Rightarrow \quad \dot{V} = -\dot{x}h(\dot{x}) - \dot{x}\varphi_D(\dot{x})$

LaSalle: V p.d., $\dot{V} \leq 0 \Longrightarrow M = \{x = 0, \dot{x} = 0\}$ is maximum invariant set.

What if I choose: $u = -\varphi_P(x)$

Robot manipulator – Example revisited with Lyapunov-based design

Dynamic model of the robotic arm:

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) = u, \qquad \theta \in R^n$$

Called *fully* actuated if n indep. actuators,

$M(\theta)$	$n \times n$ inertia matrix, $M = M^T > 0$
$C(\dot{\theta}, \theta)\dot{\theta}$	$n \times 1$ vector of centrifugal and Coriolis forces
$G(\theta)$	$n \times 1$ vector of gravitation terms

Design a controller so that $\theta \longrightarrow \theta_r$.

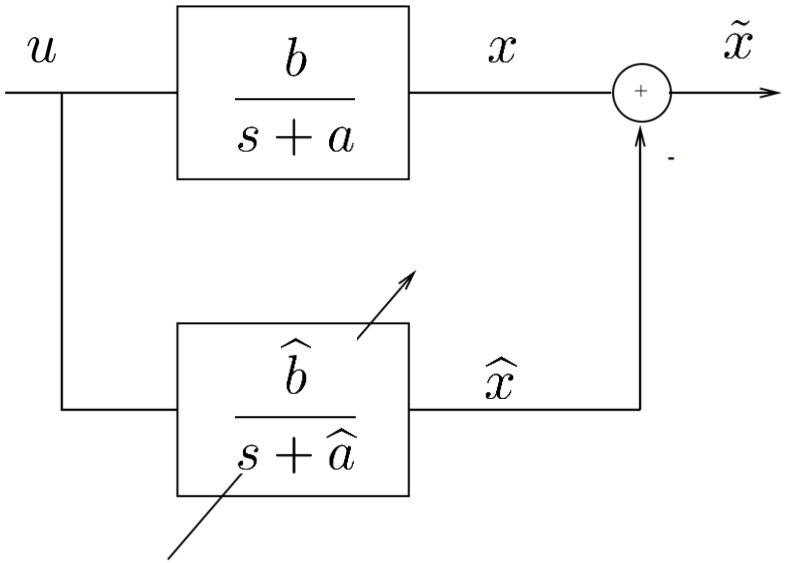
Inverse dynamics approach.

Another notable property:

$$S(\theta, \dot{\theta}) := \dot{M}(\theta) - 2C(\dot{\theta}, \theta) = -S^T(\theta, \dot{\theta})$$

Adaptive noise cancellation

$$\begin{cases} \dot{x} + ax &= bu \\ \dot{\widehat{x}} + \widehat{a}\widehat{x} &= \widehat{b}u \end{cases}$$. Design adaptation law so that $\widetilde{x} := x - \widehat{x} \to 0$



Adaptation laws or update laws: $\dot{\widehat{a}} = \ldots$, $\dot{\widehat{b}} = \ldots$

Introduce $\widetilde{x} = x - \widehat{x}$, $\widetilde{a} = a - \widehat{a}$, $\widetilde{b} = b - \widehat{b}$.
What are the dynamics of the error?

Let us try the Lyapunov function $\begin{cases} V = \frac{1}{2}(\widetilde{x}^2 + \gamma_a \widetilde{a}^2 + \gamma_b \widetilde{b}^2) \\ \dot{V} = \end{cases}$

What do we prove if $\dot{V} \leq 0$?

Are $\widetilde{a}$ and $\widetilde{b}$ proved to converge?

Simplified Adaptive control

$$\begin{cases} \dot{x} &= \theta x^2 + u \\ u &= -\hat{\theta}(t)x^2 + v \end{cases}$$

Set $\hat{\theta}(t) = \theta$

What principle of design is used?

Design:

- an update law for $\hat{\theta}$, $\dot{\hat{\theta}} = ...$
- a control signal $v(x)$

such that $x \to 0$

Introduce the new state $\tilde{\theta} = \theta - \hat{\theta}$.
Find $\dot{x} = f(x, \tilde{\theta}, v)$

Let us try the Lyapunov function $\begin{cases} V = \frac{1}{2}(x^2 + \gamma\tilde{\theta}^2) \\ \dot{V} = \end{cases}$

What do we prove if $\dot{V} \leq 0$?