FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 8: Input-Output Stability
Intro to Control-design

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Outline

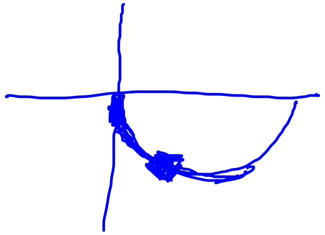
- Circle criterion and positive real functions (passivity)
- Control design based on linearization
- Lyapunov-based control design

(Strictly) Positive Real Transfer Functions

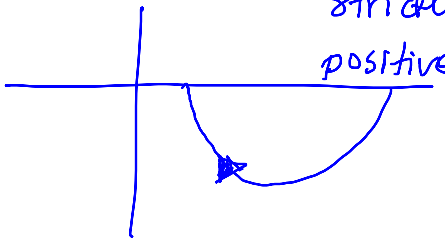
closed $[0, 1]$ open $(0, 1)$

A proper rational transfer function ~~matrix~~ $G(s)$ is positive real if:

- The poles of $G(s)$ are in $\mathrm{Re} \leq 0$
- $\mathrm{Re}[G(j\omega)] \geq 0, \forall \omega \in [0, \infty) \longrightarrow$ Nyquist plot of of $G(j\omega)$ lies in the closed right-half complex plane.



$G(s)$ is called strictly positive real if $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$
It is easier to check directly the following conditions:



- The poles of $G(s)$ are in $\mathrm{Re} < 0$.
- $\mathrm{Re}[G(j\omega)] > 0, \forall \omega \in [0, \infty)$ and $G(\infty) > 0$ or $\boxed{\lim_{\omega \to \infty} \omega^2 \mathrm{Re}[G(j\omega)] > 0}$ $\longrightarrow$ Nyquist plot of $G(j\omega)$ lies in the closed right-half complex plane and does not touch the Imaginery axis.

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Quiz

- $G(s) = \frac{1}{s}$ $\Rightarrow$ $G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega} \Rightarrow \mathrm{Re}(G(j\omega)) = 0$
- $G(s) = \frac{1}{s+1}$ $\Rightarrow$ $G(j\omega) = \frac{1}{j\omega+1} = \frac{1-j\omega}{1+\omega^2} \Rightarrow \mathrm{Re}[G(j\omega)] = \frac{1}{1+\omega^2} > 0$
- $G(s) = \frac{1}{s^2+2s+1}$

$$G(j\omega) = \frac{1}{-\omega^2+2j\omega+1} = \frac{1}{(1-\omega^2)+2j\omega} = $$

$$= \frac{1}{\omega^2\left(\frac{1}{\omega^2}+1\right)}$$

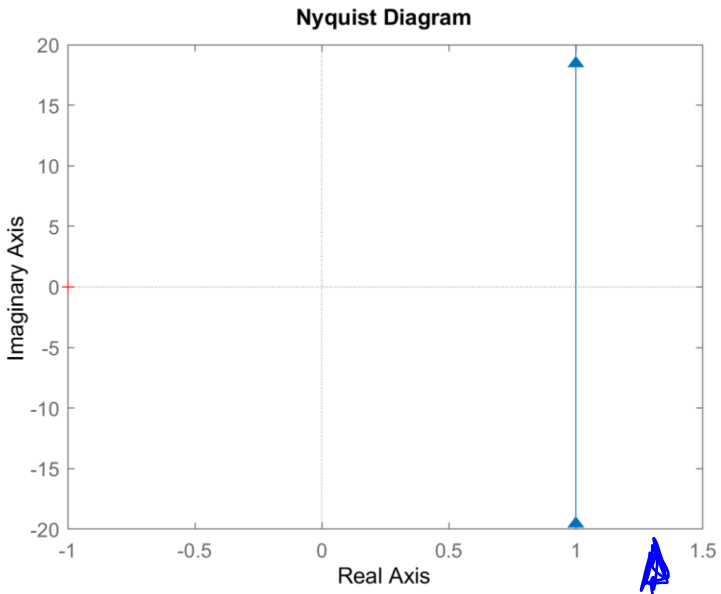
$$G(\infty) = \frac{\cancel{\omega^2}\,1}{\cancel{\omega^2}\left(\frac{1}{\omega^2}+1\right)} \to 0$$

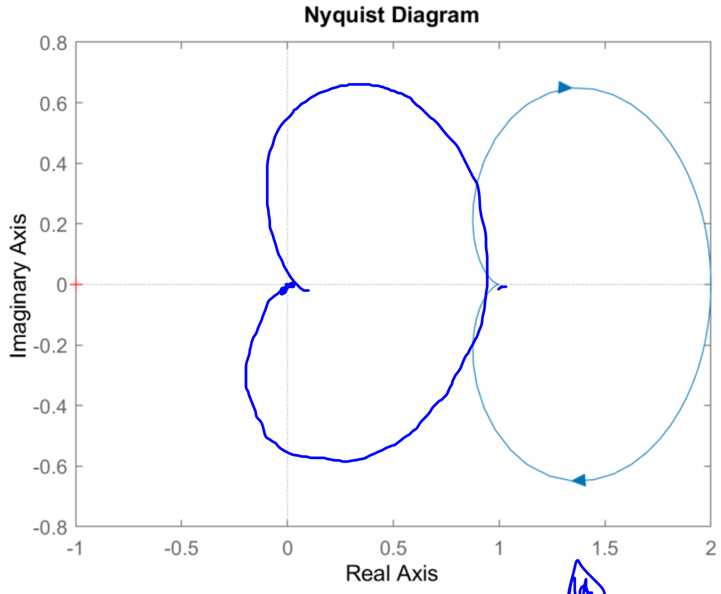
$$= \frac{(1-\omega^2)-2j\omega}{(1-\omega^2)^2+4\omega^2}$$

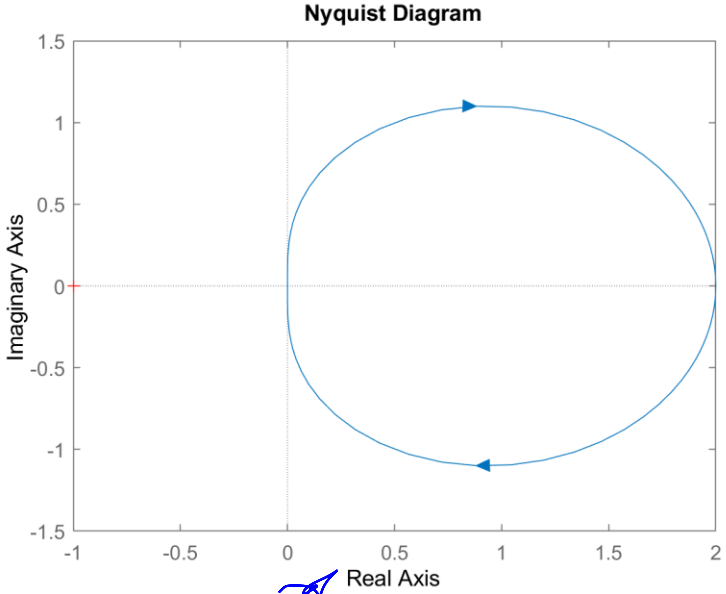
for $|\omega| > 1$ $\mathrm{Re}(j\omega) < 0$

Not positive real

Matching Quiz







a. $G(s) = \frac{s+1}{s} = 1 + \frac{1}{s}$ (integrator + 1)

b. $G(s) = \frac{s+2}{(s+1)^2}$

c. $G(s) = \frac{s^2+2s+2}{s^2+2s+1} = \frac{1+(s+1)^2}{(s+1)^2} = \boxed{1 + \frac{1}{(s+1)^2}} = 1 +$ (Non positive real transfer function)

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Kalman Yakubovich Popov Lemma

Minimal realization of $G(s)$

$$G(s) = C(sI - A)^{-1}B + D$$

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}}$$

$$R = [B \quad AB \cdots A^{n-1}B]$$

- (A, B): controllable
- (A, C): observable

$$S = \begin{bmatrix} C \\ AC \\ \vdots \\ A^{n-1}C \end{bmatrix}$$

$G(s)$ strictly positive real $\quad \exists P = P^T > 0,\ Q = Q^T \geq 0,\ \epsilon > 0$ s.t. $\begin{cases} PA + A^TP &= -Q - \epsilon P \\ PB &= C^T \end{cases}$

$G(s)$ positive real $\quad \exists P = P^T > 0,\ Q = Q^T \geq 0$ s.t. $\begin{cases} PA + A^TP &= -Q \\ PB &= C^T \end{cases}$

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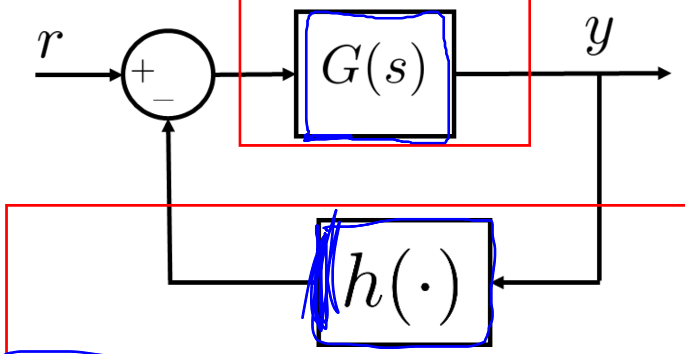
Passivity Theorem(s)

Passive Linear System

(Strictly) positive real



$h \in \text{sector}[0, \infty]$ (first and third quadrant)

- Feedback interconnection of passive systems is passive
- A passive system is BIBO
- If the input r=0, the origin of minimum realization of G(s) is gas

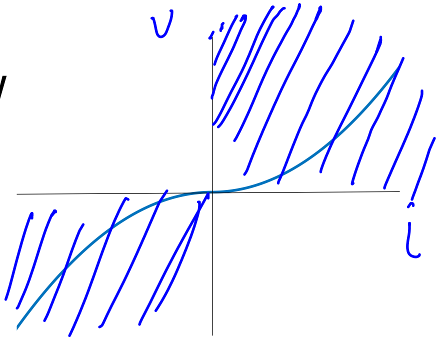
- $h \in [0, \infty]$ and $Z(s) = G(s)$ is strictly positive real

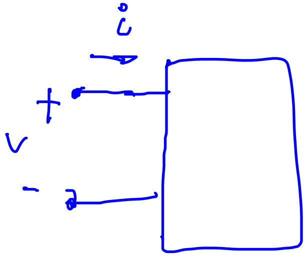
The feedback system is BIBO

Power inflow

$p = vi$

$p = Fv$





passive network

$p(t) = v(t)\, i(t)$

$\underbrace{E(t)}_{\text{energy absorbed}} = E(0) + \int_0^t p(t)$

if $p(t) > 0$ box absorbs energy

if $p(t) < 0$ box delivers energy

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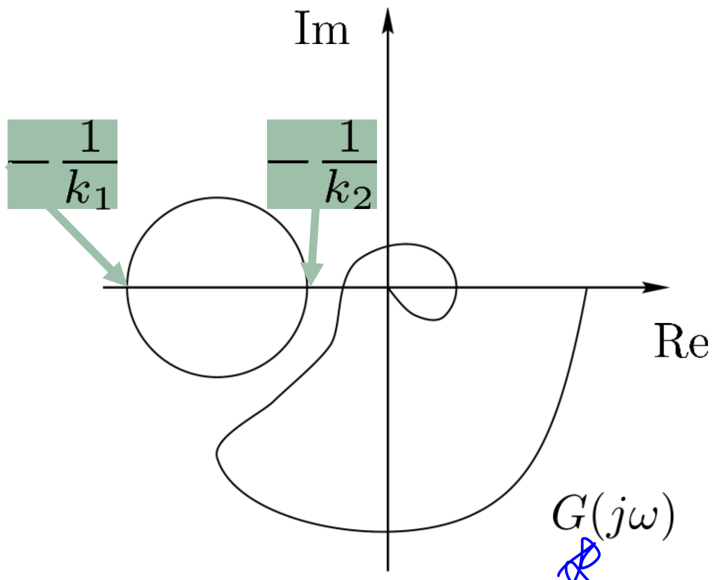
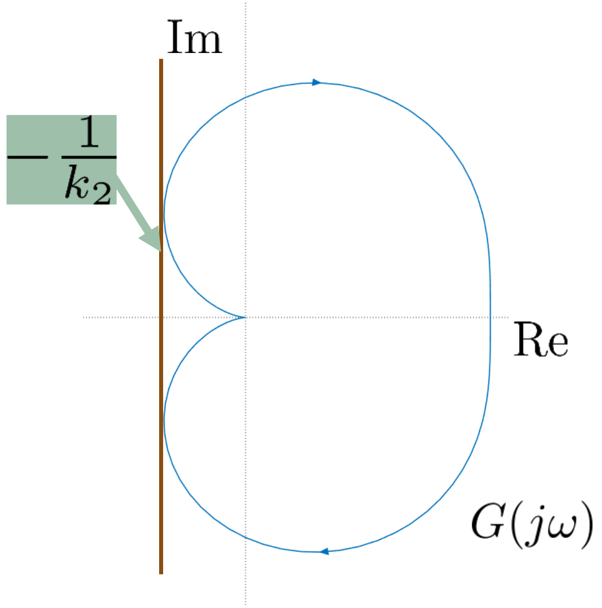
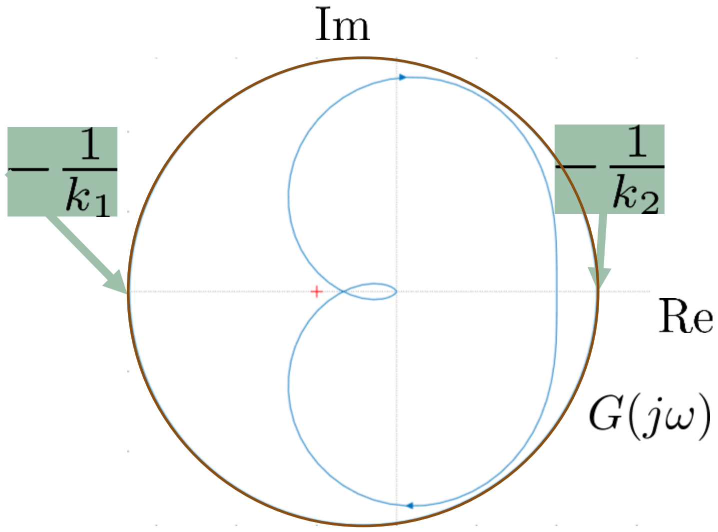
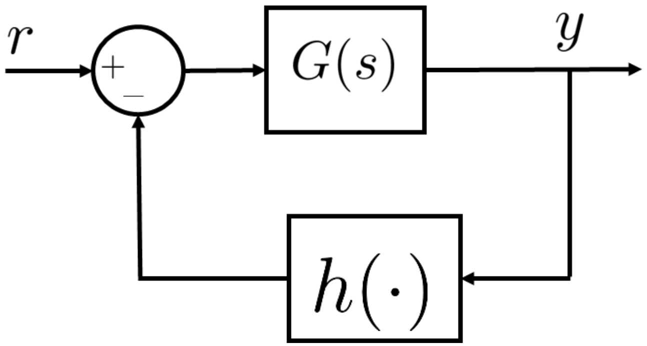
Passivity theorem – Circle criterion









- $h \in [0, \infty]$ and $Z(s) = G(s)$ is strictly positive real
- $h \in [k_1, \infty]$ and $Z(s) = \frac{G(s)}{1+k_1 G(s)}$ is strictly positive real

$0 < k_1 < k_2$

- $h \in [k_1, k_2]$ and $\boxed{Z(s) = \frac{1+k_2 G(s)}{1+k_1 G(s)}}$ is strictly positive real

The system is BIBO and if $r = 0$ the origin GAS

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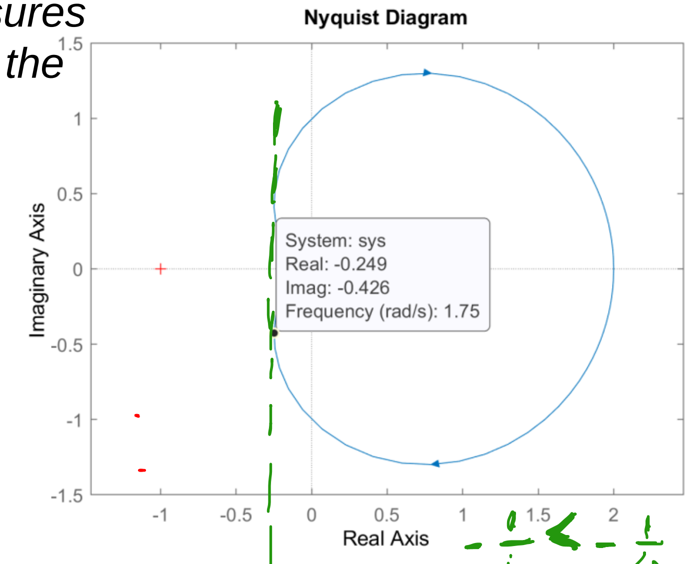
Circle criterion – Strictly positive real functions

$$\begin{cases} \dot{x}_1 = -x_1 - \operatorname{sign}(x_1)\min(K|x_1|, 1) \\ \dot{x}_2 = -x_2 + 2x_1 \end{cases}$$

Find the maximum value of K that ensures asymptotic stability of the origin using the circle criterion.



$-\frac{1}{K} < -\frac{1}{4} \Rightarrow$

1) Isolate the linear part of the system (A). Check the argument (input) of the nonlinear function (c). Check how the nonlinearity is involved in the system(b). Find G(s) using A, b, c.

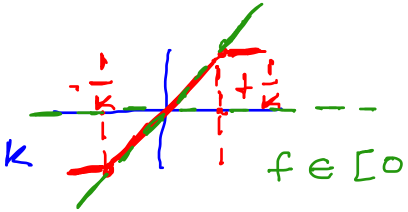
$$\begin{cases} \dot{x} = Ax + bu \\ y = cx \end{cases}$$

$A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $c = [1 \quad 0]$

2) Nonlinearity: Find the sector as a function of the gain K

$\operatorname{sign}(x_1)\min(K|x_1|, 1)$ if $K|x_1| < 1$ $f(x) = Kx_1$

if $K|x_1| > 1$ $f(x) = \operatorname{sgn}(x_1) K$



$f \in [0 \; K]$

3) Check if the Z(s) is strictly positive real

if $\Delta < 0$ then the quadratic polynomial is positive for any $x = \omega^2$

$$G(j\omega) = \frac{2}{1-\omega^2+2j\omega} = \frac{2(1-\omega^2)}{(1-\omega^2)^2+4\omega^2} - j\frac{4\omega}{(1-\omega^2)^2+4\omega^2}$$

Check the discriminant (*) of the quadratic equation $x^2 + 2(1-K)x + 2K+1 = 0$

$$\operatorname{Re}[1 + KG(j\omega)] = 1 + K\frac{2(1-\omega^2)}{(1-\omega^2)^2+4\omega^2} = \frac{\boxed{\omega^4+2(1-K)\omega^2+2K+1}}{(1-\omega^2)^2+4\omega^2} \Rightarrow K < 4$$

$\Delta = 2(1-K)^2 - 4(2K+1)$ (*)

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Linear control design based on linearization

$$\dot{x} = f(x, u)$$

$$\frac{d\tilde{x}}{dt} = A\tilde{x} + B\tilde{u}$$

$$A = \frac{\partial f}{\partial x}(x^\star, u^\star) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{x=x^\star, u=u^\star}$$

$$B = \frac{\partial f}{\partial u}(x^\star, u^\star) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix}\Bigg|_{x=x^\star, u=u^\star}$$

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(A, B) is not controllable

Controllability condition: $\mathrm{rank}(R) = n$

$$R = [B \quad AB \cdots A^{n-1}B]$$

$$\dot{x} = Ax + Bu$$

$$u = -Kx$$

CLS $= (A - BK)\cancel{x}$

$$\dot{x} = f(x, u^\star - K(x - x^\star))$$

$$u = u^\star - \boxed{K(x - x^\star)}$$

$$\dot{\tilde{x}} = (A - BK)\tilde{x}$$

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Example (Linearization)

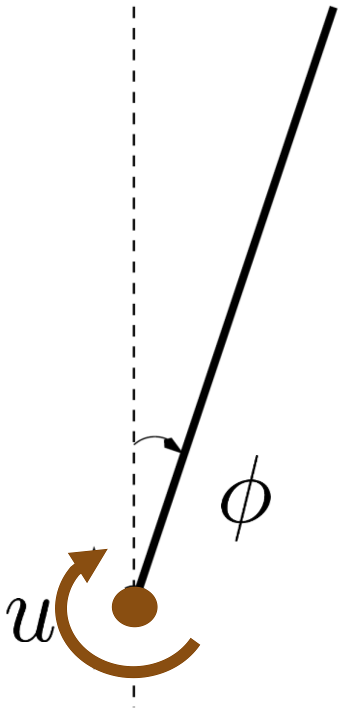
An inverted pendulum controlled by a motor torque u at the joint:

$$\ddot{\phi}(t) = \frac{g}{l}\sin(\phi(t)) + \frac{1}{ml^2}u,$$

where $u(t)$ is acceleration, can be written as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g}{l}\sin(x_1) - \frac{1}{ml^2}u \equiv 0\end{aligned}$$

$$x^* = \begin{array}{c} \begin{bmatrix} \delta \\ 0 \end{bmatrix} \end{array} \begin{array}{c} \phi \\ \dot{\phi} \end{array} \qquad u^* = mgl\sin(\delta)$$



Linearize the system and find a control input that can stabilize the system at angle δ? Is the linear system controllable for all δ?

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos(x_1) & 0 \end{bmatrix}\Bigg|_{\substack{x=x^* \\ u=u^*}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos\delta & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & -\frac{1}{ml^2} \\ -\frac{1}{ml^2} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -\frac{1}{ml^2} \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos\delta & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -\frac{1}{ml^2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{ml^2} \\ 0 \end{bmatrix}$$

$$u = mgl\sin\delta - k_1\dot{\phi} - k_2(\phi - \delta)$$

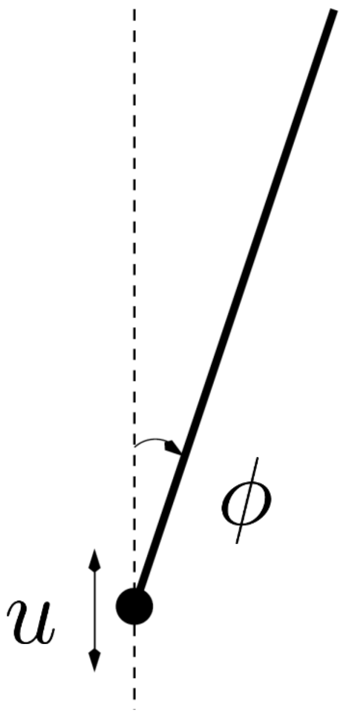
Example (Linearization)

An inverted pendulum with vertically moving pivot point

$$\ddot{\phi}(t) = \frac{1}{l}(g + u(t))\sin(\phi(t)),$$

where $u(t)$ is acceleration, can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{l}(g+u)\sin(x_1) \end{aligned}$$



Try this home – See lecture notes 2.

Lyapunov-based design

Steps of Lyapunov-based design:

$\dot{x} = f(x, u)$

1. Select a positive definite $V(x)$.
2. Calculate $\dot{V}(x) = \boxed{\frac{\partial V}{\partial x}} f(x, u)$.
3. Find a (possibly) nonlinear feedback control law that makes $\dot{V}$ negative.

$\frac{\partial V}{\partial x} f(x, u) \leftarrow 0$

- $\dot{V} \leq 0 \longrightarrow x = 0$ may be asymptotically stable (check LaSalle)
- $\dot{V} < 0$ for all $x \neq 0 \longrightarrow x = 0$ asymptotically stable
- $\dot{V} \leq -\lambda V \longrightarrow x = 0$ exponentially stable if addtionally $V \geq c\|x\|^2$

Comments:

- Selection of $V(x)$
- Depends on the system dynamics $\dot{x} = f(x, u)$

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Example 1 (Lyapunov-based design)

Consider the nonlinear system

$$\dot{x}_1 = -3x_1 + 2x_1x_2^2 + u$$
$$\dot{x}_2 = -x_2^3 - x_2,$$

Find a nonlinear feedback control law which makes the origin globally asymptotically stable.

$$\dot{V} = -3x_1^2 - x_2^2 - x_2^4$$

$$\dot{V} = -x_1^2 - x_2^2 \quad -2x_1^2 - x_2^4$$

$$\dot{V} \le -x_1^2 - x_2^2 = -2V \Rightarrow \boxed{\dot{V} \le -2V}$$

$$V = \frac{1}{2}(x_1^2 + x_2^2)$$

$$\dot{V} = [x_1 \quad x_2]\begin{bmatrix} -3x_1 + 2x_1x_2^2 + u \\ -x_2^3 - x_2 \end{bmatrix} = \underbrace{-3x_1^2}_{<} + \underbrace{2x_1^2x_2^2}_{>0} + x_1u \underbrace{- x_2^4}_{<0} - \underbrace{x_2^2}_{<0}$$

$$u = -2x_1x_2^2$$

$$= -3x_1^2 + \cancel{2x_1^2x_2^2} - \cancel{2x_1^2x_2^2} - x_2^4 - x_2^2$$

$\dot{V} < 0$ for all $x \neq 0$

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Example 2 (Lyapunov-based design)

Consider the nonlinear system

$$\begin{aligned}
\dot{x}_1 &= -3x_1 + 2x_1x_2^2 + u \\
\dot{x}_2 &= -x_2^3 - x_2,
\end{aligned}$$

Find a nonlinear feedback control law which makes the origin globally exponentially stable.

Example 3 (Lyapunov-based design)

Consider the system

$$\dot{x}_1 = x_2^3$$
$$\dot{x}_2 = u$$

Find a globally asymptotically stabilizing control law $u = u(x)$.