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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 6: Lyapunov stability II

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Stability of an equilibrium point of a nonlinear system – Lyapunov's Direct Method

$\dot{x} = f(x),\ f(x^\star) = 0 \qquad x^\star = 0 \in \Re^n$

$\Omega(\Re^n)$

$\exists\, V : \Omega(\Re^n) \to \Re$

$V(x)$ Positive definite

$V(x^\star) = 0$
$V(x) > 0 \;\forall\, x \neq x^\star$

$\boxed{V(x) \quad \text{Radially unbounded}}$

$V(x) \to \infty$ as $\|x\| \to \infty$

$-\dot{V}(x)$ Positive semidefinite

$-\dot{V}(x) \geq 0$

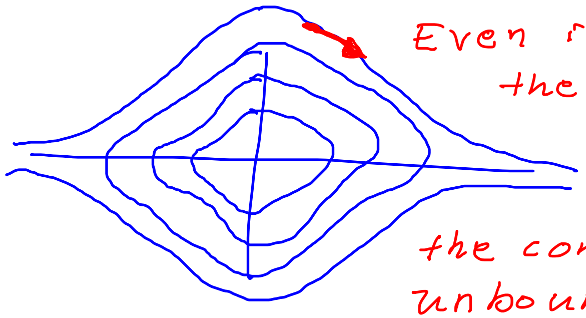
↳ It defines a set of equilibria

→ **Stable Equilibrium**

$-\dot{V}(x)$ Positive definite

Stable Equilibrium + $-\dot{V}(x)$ Positive definite → **Asymptotically Stable Equilibrium**

$\Omega(\Re^n)$



Even if $\dot{V}(x) = 0$ the trajectory can escape to infinity if the condition on radially unboundedness is not satisfied

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Lyapunov stability analysis - comments

- The conditions of the Theorem are only sufficient

If conditions are not satisfied:

It does not mean that the equilibrium is unstable.

It means that the chosen Lyapunov function does not allow to make a conclusion

It requires further investigation

- ✓ *try to find another Lyapunov function*
- ✓ *Use other Theorems* ☺

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Outline

- Softer conditions
- Convergence rate (exponential stability)
- Invariant Sets
- Region of attraction
- Asymptotic stability of invariant sets
- Lyapunov stability for linear systems

Asymptotic Stability (softer condition on $\dot{V}$)

Barbashin, Krasovskii Theorem (LaSalle Invariance Principle is more general and proved afterwards, we can call this LaSalle Theorem)

Theorem: Let $\dot{x} = f(x)$ and $f(x^\star) = 0$. If there exists a $\mathcal{C}^1$ function $V : \Re^n \to \Re$ such that

we call this LaSalle Theorem and we use it when

(1) $V(x) > 0$ for all $x \neq x^\star$ and $V(x^*) = 0$

(2) $V(x) \to \infty$ as $\|x\| \to \infty$

(3) $\boxed{\dot{V}(x) \leq 0 \text{ for all } x}$

1. Find the solution correspo-nding to $\dot{V} \equiv 0$
2. Substitute in $\dot{x} = f(x)$ and show that corresponds to $x = x^\star$

(4) No solution of $\dot{x} = f(x)$ can stay identically in $E = \left\{x \in \Re^n : \dot{V} = 0\right\}$ except of $x = x^\star$

then $x^\star$ is globally **asymptotically** stable.

– $\dot{V}$ is not positive definite

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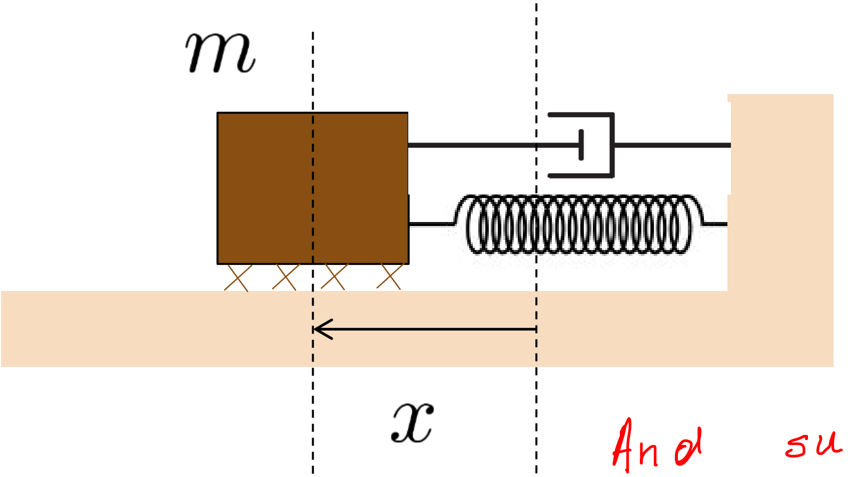
Example (revisited)

$$m\ddot{x} = -b\dot{x}|\dot{x}| - k_0x - k_1x^3 \quad (*)$$



$$V(x,\dot{x}) = (\underbrace{2m\dot{x}^2}_{\text{kinetic}} + \underbrace{2k_0x^2 + k_1x^4}_{\text{potential energy}})/4 > 0, \qquad V(0,0) = 0$$

$\dot{V}(x,\dot{x}) = -b|\dot{x}|^3$ gives $E = \{(x,\dot{x}) : \dot{x} = 0\}$.

- $\dot{V} \le 0$ *(it is not negative definite because it can be zero for all x)*
  negative semidefinite
- $\dot{V} \equiv 0 \Rightarrow \dot{x} \equiv 0 \overset{\ddot{x}=0}{\Longrightarrow} k_0x(1 + \frac{k_1}{k_0}x^2) \equiv 0 \Rightarrow x_1 = 0$

$\dot{V} \equiv 0$ implies $\dot{x} \equiv 0$
Differentiating we get $\ddot{x} \equiv 0$.
And substituting in $(*)$ we get $x \equiv 0$

⬇ *Barbashin, Krasovskii or LaSalle*

- Global asymptotic stability of $(x,\dot{x}) = (0,0)$

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Exponential stability

Theorem: Let $\dot{x} = f(x)$ and $f(x^\star) = 0$. If there exists a $\mathcal{C}^1$ function $V : \Re^n \to \Re$ and numbers $\alpha, \epsilon, c > 0$ such that

More strict condition than p.d.

(1) $V(x^\star) = 0$

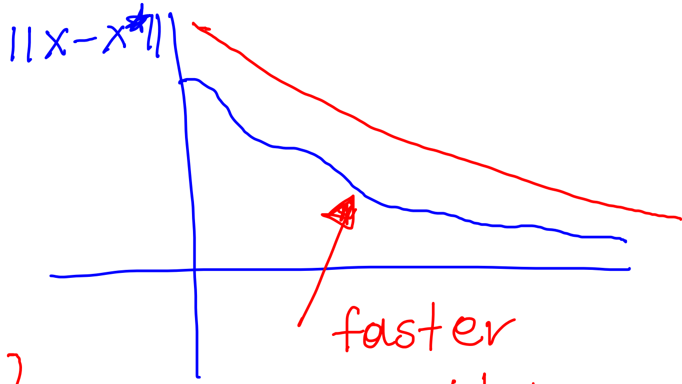
(2) $V(x) > \epsilon \|x - x^\star\|^c > 0$ for all $x \neq x^*$

(3) $\dot{V}(x) \leq -\alpha V(x)$ for all x

(4) $V(x) \to \infty$ as $\|x\| \to \infty$

then $x^\star$ is globally **exponentially** stable.

radially unbounded



faster than $-\frac{\alpha}{c}$

From (3)

$V(x) \leq V(0)\, e^{-\alpha t}$

From (2)

$\epsilon \|x - x^\star\|^c \leq V(0)\, e^{-\alpha t}$

$\Rightarrow \|x - x^\star\| \leq \left(\frac{V(0)}{\epsilon}\right)^{\frac{1}{c}} e^{-\frac{\alpha}{c} t}$

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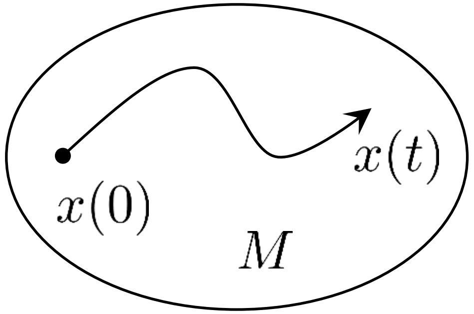
Invariants Sets

Invariant set M for the system $\dot{x} = f(x)$.

$$x(0) \in M \Longrightarrow x(t) \in M \ \forall t \geq 0.$$



Examples:

- equilibrium points
- limit cycles
- the whole $\mathbb{R}^n$
- Lyapunov sets

Lyapunov sets as invariant sets

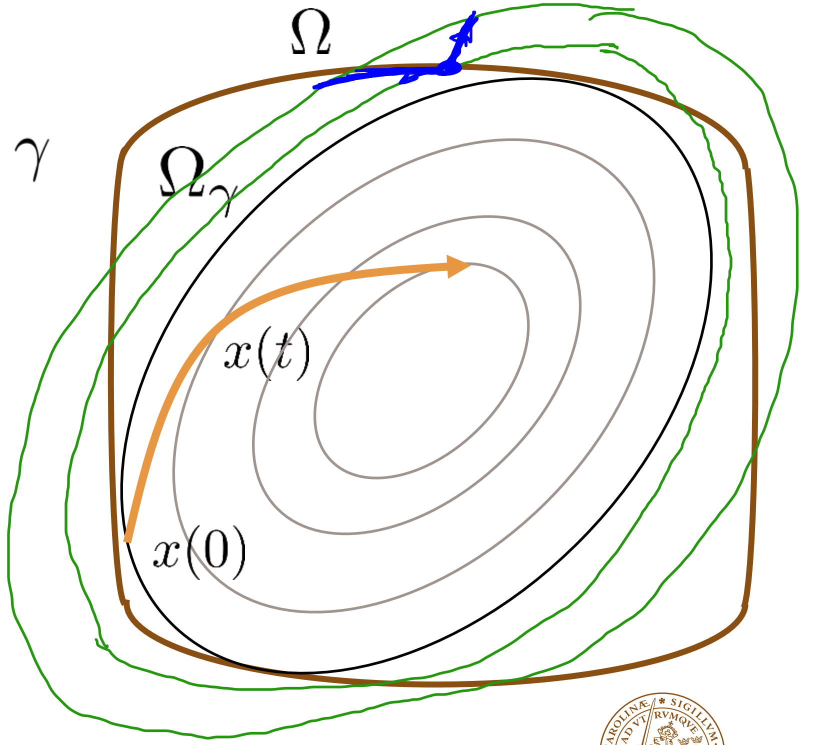
- Notice that the condition $\dot{V} \leq 0$ implies that if a trajectory crosses a Lyapunov surface $V(x) = \gamma$ it can never come out again.

see next slides for explanation on why $\underline{\Omega}$ is not invariant

$\dot{f} \leq 0 \Rightarrow f(t_2) \leq f(t_1)$

Why? $\boxed{x(0) \in \Omega_\gamma}$ $\quad \dot{V} \leq 0 \Longrightarrow \quad V(x(t)) \leq V(x(0)) \leq \gamma$

$V(x(0)) \leq \gamma$ $\quad x(t) \in \Omega_\gamma = \{x \in \Re : \; V(x) \leq \gamma\}, \forall t \geq 0$



- If $V(x) \in \mathcal{C}^1$ and satisfies $\dot{V}(x) \leq 0$ along the solutions of $\dot{x} = f(x)$, then the set:

$$\Omega_\gamma = \{x \in \Re : \; V(x) \leq \gamma\} \subset \Omega$$

is an invariant set.

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Region of Attraction

- Local asymptotic stability theorems guarantee existence of a possibly small neighborhood of the equilibrium point where such an attraction takes place

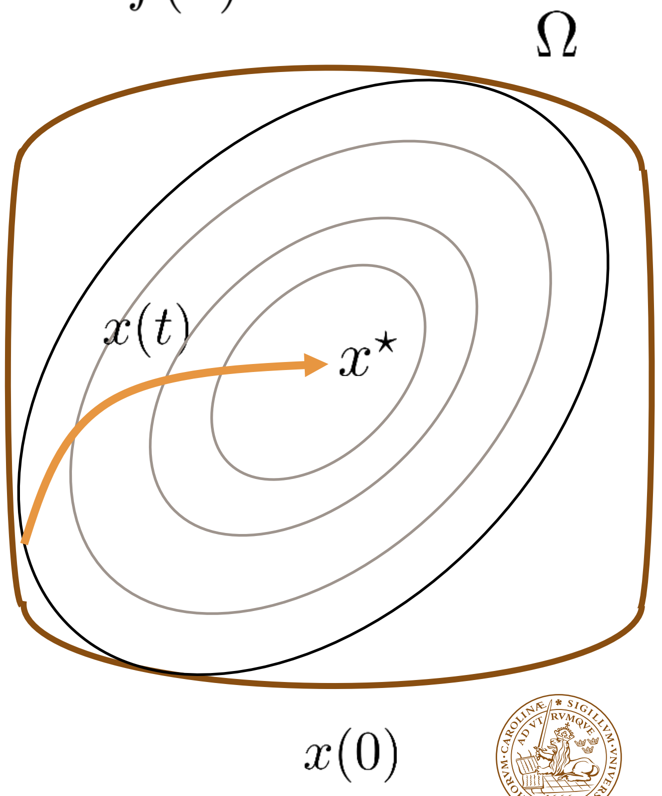
- The region of attraction to the equilibrium point $x^\star$ of the system $\dot{x} = f(x)$ is defined by $\mathcal{R}_A = \{x(0) \in \Omega : x(t) \to x^\star \text{ as } t \to \infty\}$.

- If $V(x) \in \mathcal{C}^1$ and satisfies $\dot{V}(x) \leq 0$ along the solutions of $\dot{x} = f(x)$, then the set:

$$\Omega_\gamma = \{x \in \Re : \; V(x) \leq \gamma\} \subset \Omega$$

  is an invariant set and can be used as an **estimate of region of attraction**.

- The estimate of the region of attraction based on Lyapunov level sets is conservative $\Omega_\gamma \subset \mathcal{R}_A$



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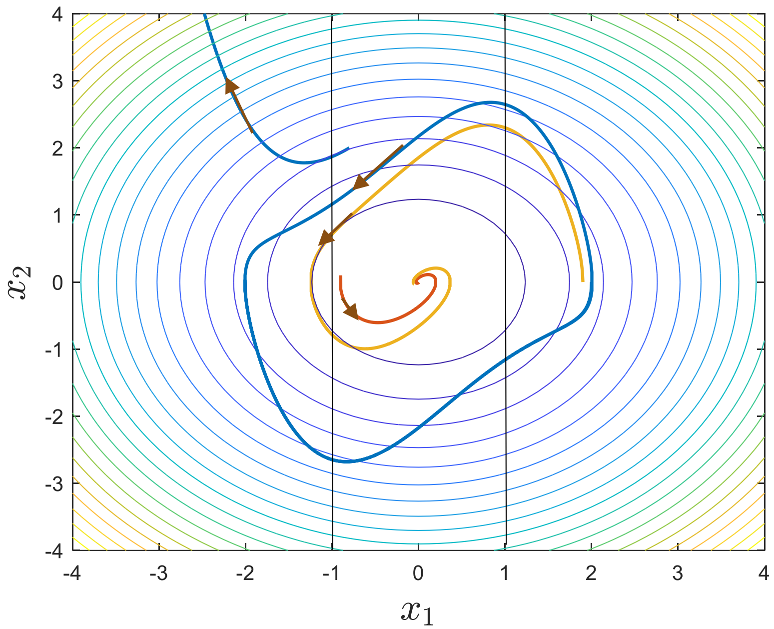
Discussion

Why we cannot claim that Ω is an estimate of region of attraction?



$$\Omega = \{x \in \Re^n : \quad \dot{V} \leq 0\}$$

Van der Pol equation in reverse time

$$\dot{x}_1 = -x_2$$

$$\dot{x}_1 = x_1 - (1 - x_1^2)x_2$$

$V = \frac{1}{2}(x_1^2 + x_2^2)$ positive definite for all x

$\dot{V} = -(1 - x_1^2)x_2^2$ negative semidefinite for $|x_2| < 1$

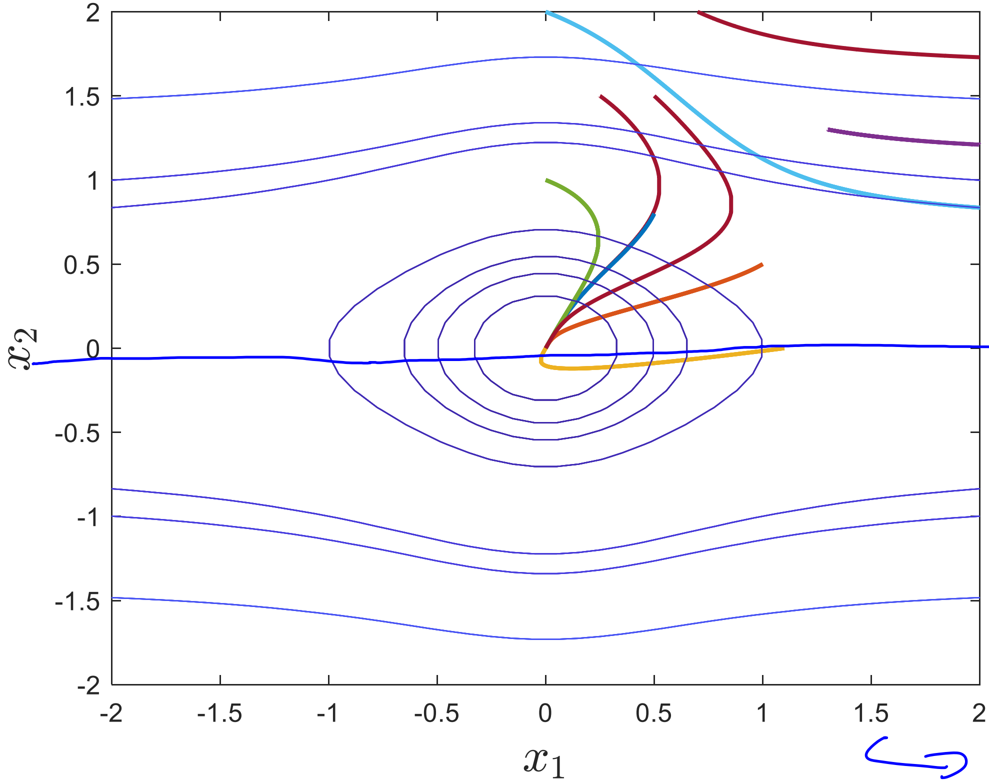
The conditions for applying LaSalle Theorem for asymptotic stability are satisfied in $\Omega = \{|x_1| < 1, |x_2| < L\}$ L arbitrarily large constant.

Thus, the origin is local asymptotically stable. However Ω is not invariant. Starting within Ω the trajectory can move to Lyapunov surfaces $V(x) = \gamma$ with smaller γs but there is no guarantee that the trajectory will remain in Ω. See the blue trajectories. Once leaving Ω, $\dot{V}$ could be positive and the trajectory may move to Lyapunov surfaces with higher γ. Observe that one of the blue trajectories is a limit cycle. Charecterize its stability.

Example 1

Example:

$$\begin{cases} \dot{x}_1 = \frac{-6x_1}{(1+x_1^2)^2} + 2x_2 \\ \dot{x}_2 = \frac{-2(x_1+x_2)}{(1+x_1^2)^2} \end{cases}$$



$$\begin{cases} x_2 = 0 \\ \frac{1}{2}\frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2 = c \end{cases} \Rightarrow \left\{ \frac{1}{2}\frac{x_1^2}{1+x_1^2} = c \right.$$

$\Rightarrow (1-2c)\, x_1^2 = 2c$

It has solution when $c < \frac{1}{2}$ (*)

- $V = \frac{1}{2}\frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2 > 0$ and $V(0) = 0$
- $V = \frac{1}{2}\frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2$ is radially unbounded for $V(x) < 1$
- $\dot{V} = \frac{x_1\dot{x}_1}{(1+x_1^2)^2} + x_2\dot{x}_2 = -6\frac{x_1^2}{(1+x_1^2)^4} - 2\frac{x_2^2}{(1+x_1^2)^2} < 0,\ \dot{V}(0) = 0$

$x_2 = 0$

The axis $x_2 = 0$ intersects with a Lyapunov surface $V(x) = c$ when $c < \frac{1}{2}$

$\Leftrightarrow V(x) = c$ is closed

- Local asymptotic stability
- Estimate of region of attraction $\Omega_{1/2} = x \in \Re^2 : V(x) < 1/2$

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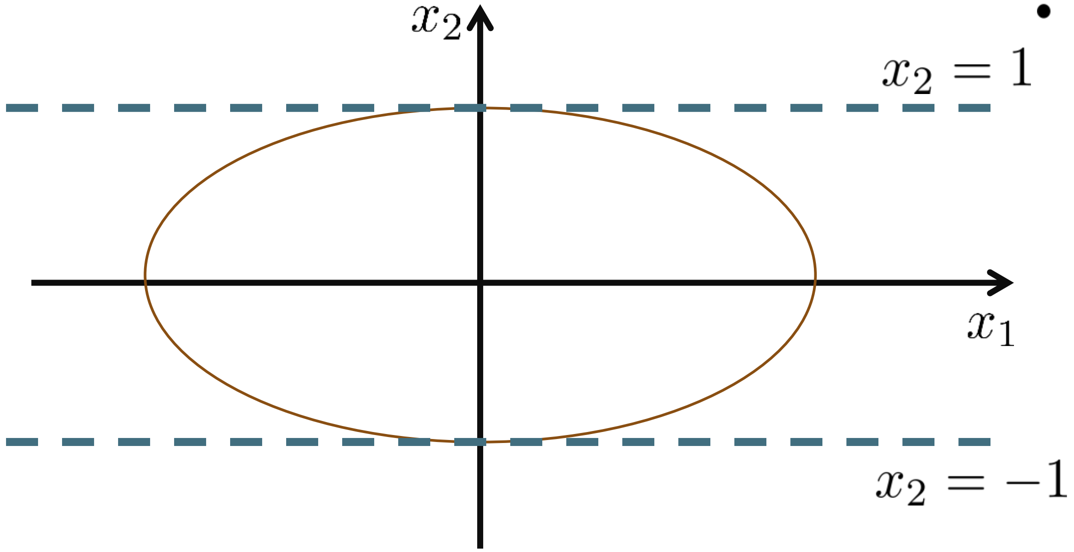
Example 2

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = \frac{x_1}{2} + x_2^3 - x_2 \end{cases}$$

Eq. Point

$$\begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = 0 \\ x_1 = 0 \end{cases}$$



- $V = \frac{x_1^2}{2} + x_2^2 > 0$ and $V(0) = 0$
- $V = \frac{x_1^2}{2} + x_2^2$ radially unbounded
- $\dot{V} = x_1\dot{x}_1 + 2x_2\dot{x}_2 = \cdots = 2x_2^2(x_2^2 - 1)$
  - $\dot{V} \leq 0$ for $|x_2| < 1$
  - $\dot{V} \equiv 0 \Rightarrow x_2 \equiv 0 \xrightarrow{\dot{x}=f(x)} x_1 \equiv 0$ for $|x_2| < 1$

$$\dot{V} = \nabla V \cdot \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = x_1\dot{x}_1 + 2x_2\dot{x}_2 = \cancel{-x_1x_2} - 2x_2^2 + 2x_2^4 \cancel{+x_1x_2}$$

$\dot{V} \equiv 0 \Rightarrow x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0$

$V(x) < 1$

LaSalle

- Local asymptotic stability
- Estimate of region of attraction $\Omega_1 = x \in \Re^2 : V(x) < 1$

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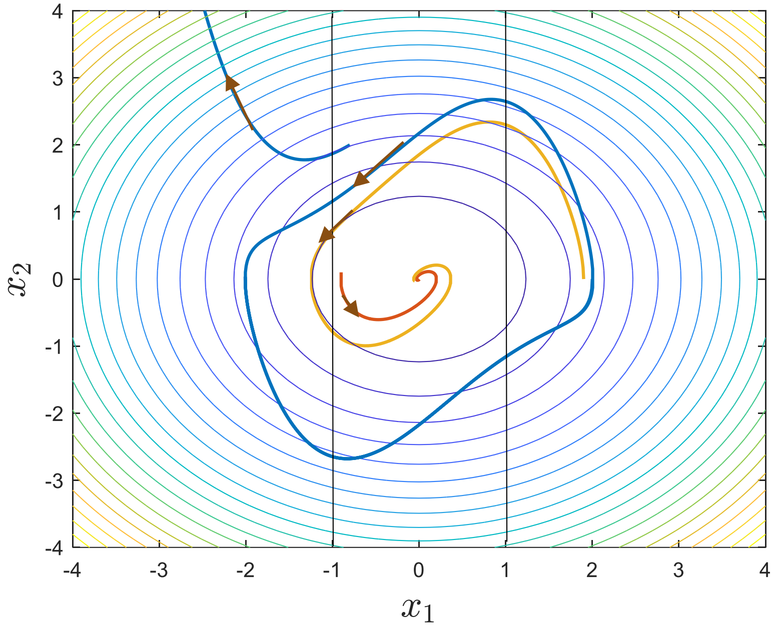
Example 3



Van der Pol equation in reverse time

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$V = \frac{1}{2}(x_1^2 + x_2^2)$ positive definite for all x

$\dot{V} = -(1 - x_1^2)x_2^2$ negative semidefinite for $|x_1| < 1$

The conditions for applying LaSalle Theorem for asymptotic stability are satisfied in $\Omega = \{|x_1| < 1, |x_2| < L\}$ L arbitrarily large constant. Thus, the origin is local asymptotically stable.

- Derive an estimate of the region of attraction.
- Which is the actual region of attraction?

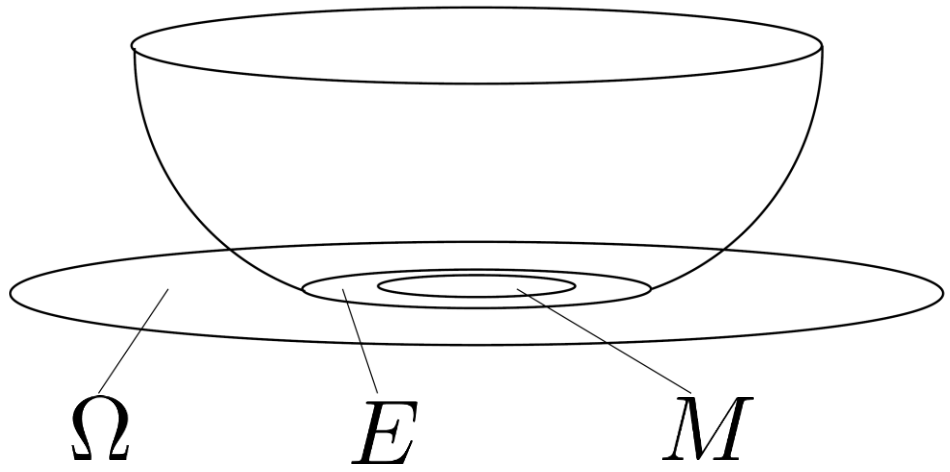
LaSalle's invariance principle

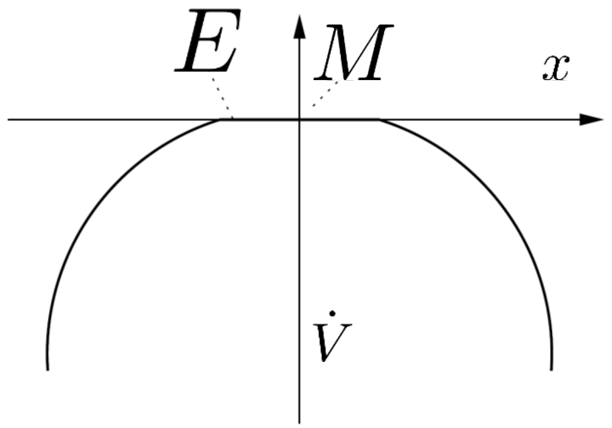
LaSalle's invariant set Theorem

- Let $\Omega \subseteq \Re^n$ compact invariant set for $\dot{x} = f(x)$.
- Let $V : \Omega \to \Re$ be a $\mathcal{C}^1$ function such that $\dot{V}(x) \leq 0, \ \forall x \in \Omega$.
- $E := \{x \in \Omega : \ \dot{V}(x) = 0\}$, M :=largest invariant subset of E

$\forall x(0) \in \Omega$, $x(t)$ approaches M as $t \to +\infty$

Note that Ω can be defined indepednent of V. In many cases, it is easier to construct Ω based on V as $\Omega = \Omega_\gamma = V(x) \leq \gamma$.





Example – Limit Cycle

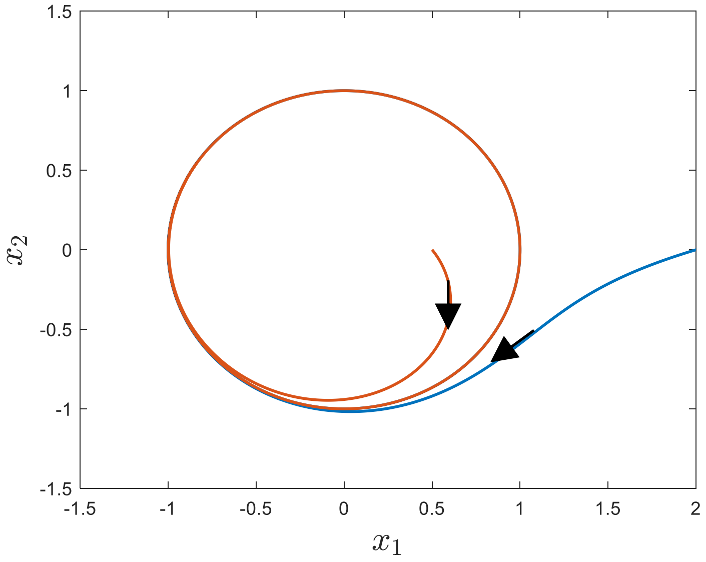
Show that $M = \{x : \|x\| = 1\}$ is a asymptotically stable limit cycle for (almost globally, except for starting at $x = 0$)

$\|x\|^2 = x_1^2 + x_2^2.$

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)$$

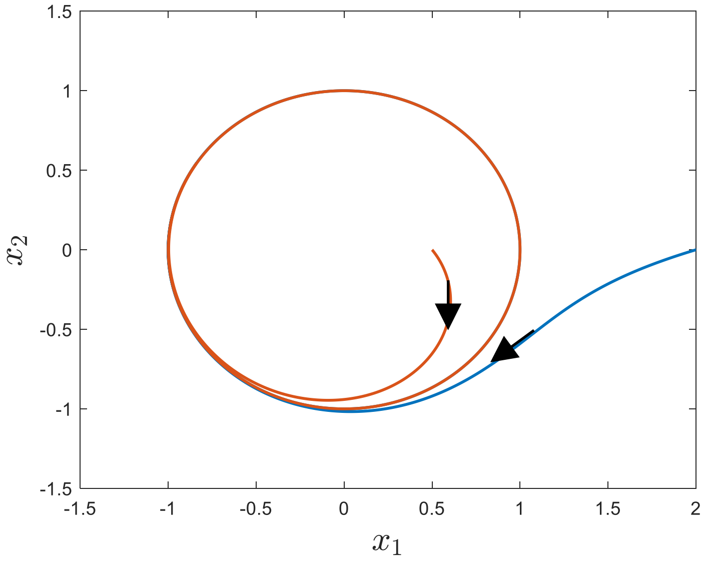
The system has one equilibrium at the origin and one limit cycle. Thus the set of trajectories that are invariant for the system are in the set $E = \{x : \|x\| = 0 \text{ or } \|x\| = 1\}$. We can actually show this by calculating the derivative $\frac{d}{dt}(\|x\|^2) = -\|x\|^2(\|x\|^2 - 1)$:

- If $\|x(0)\| = 0$ or $\|x(0)\| = 1$ the derivative is zero $\frac{d}{dt}(\|x(0)\|^2)$and thus the norm of x will not change.
- $\|x(t)\| = 0$ corresponds to the equilibrium point at the origin $\dot{x}_1 = \dot{x}_2$ while $\|x(t)\| = 1$ corresponds to $\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & -x_1 \end{cases}$ defining a limit cycle moving clockwise
- Remark: From the derivative $\frac{d}{dt}(\|x\|^2) = -2\|x\|^2(\|x\|^2 - 1)$ and if we consider a Lyapunov-like function $V_0 = \frac{1}{2}$ we can see that $\Omega' = \{\|x\| < 1\}$ cannot be proved invariant since $\dot{V}_0 > 0$.



Example – Limit Cycle

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)$$



- Take the Lyapunov-like $V = (x_1^2 + x_2^2 - 1)^2$, that is positive but not positive definite. It encodes some dinstance metric from the limit cycle $x_1^2 + x_2^2 = 1$.

- Differentiating V along the system trajectories, we get: $\dot{V} = -4(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1)^2 \leq 0$.

- Choose $\Omega = \{x \in \Re^2 : 0 < \|x\| \leq 1\}$ to exclude $\|x\| = 0$. Note that Ω is invariant as it is subset of $\Omega_1 = \{V < 1\}$. Check this. By excluding $x = 0$, the maximum invariant set is $M = \{x \in \Omega : \|x\| = 1\}$.

LaSalle's Invariance Principle

$x \to M$ as $t \to \infty$ *almost globally*

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Example – Set of equilibriums

λ constant

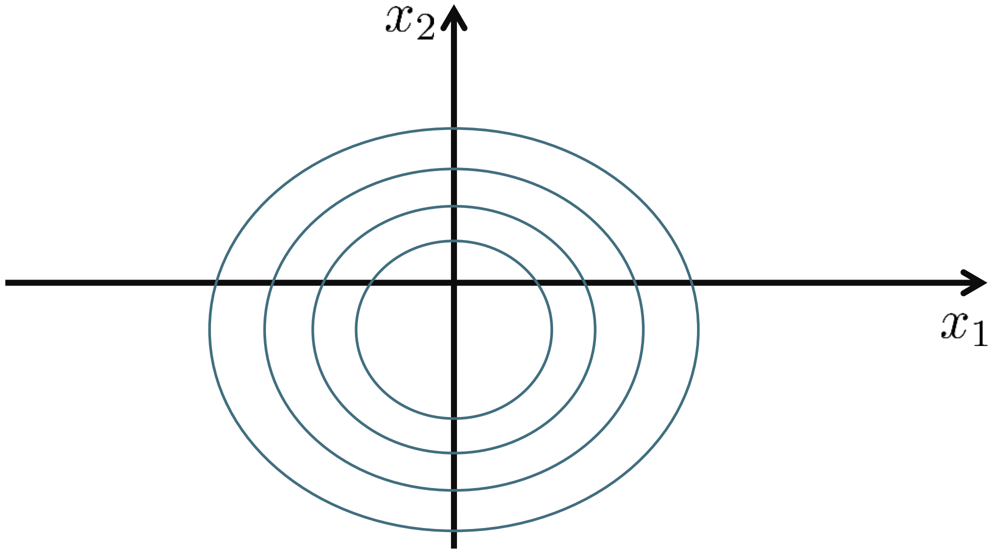
$$\begin{cases} \dot{x}_1 = \lambda x_1 - x_1 x_2 \\ \dot{x}_2 = a x_1^2 \end{cases}$$

At equilibrium

$$\begin{cases} \dot{x}_1 = \lambda x_1 - x_1 x_2 = 0 \\ \dot{x}_2 = a x_1^2 = 0 \end{cases} \Rightarrow \quad x_1 = 0,\ x_2 \in \Re$$

c constant

- $V = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - c)^2 > 0$, radially unbounded
- $\dot{V} = -(c - \lambda)x_1^2 \leq 0$ for $c > \lambda$
- $E := \{x \in \Re^2 : \dot{V}(x) = 0\} \equiv M := \{x \in \Re^2 : x_1 = 0\}$



LaSalle's Invariance Theorem

- $x \to M$ as $t \to \infty$

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Lyapunov analysis for Linear systems

Linear system: $\dot{x} = Ax$

$$\dot{x} = Ax = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To check stability:

1. Find the eigenvalues of A, λ_i.
2. Verify that they are negative.

Eigenvalues of A : $\{-1, -3\}$

$\Rightarrow$ (global) asymptotic stability.

Try to prove stability with:

$$V(x) = \frac{1}{2}\|x\|^2 = \frac{1}{2}x^T x = \frac{1}{2}(x_1^2 + x_2^2)$$

$$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2$$

$$= -x_1^2 + 4x_1x_2 - 3x_2^2$$

$$= -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\det(M) = 3 - 4 = -1$

$-\dot{V}$ is not p.d.

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Lyapunov analysis for Linear systems

Linear system: $\dot{x} = Ax$

To check stability:

1. Find the eigenvalues of A, λ_i.
2. Verify that they are negative.

Try to prove stability with:

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$V(x) = \|x\|^2 = x^T P x$$

Parametric Lyapunov function in a quadratic form

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x$$

Choose parameters for P such that $-\dot{V}(x)$ *p.d.*

$$-x^T Q x < 0$$

$\hookrightarrow$ e.g. $Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$ $q_1, q_2 > 0$

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Lyapunov analysis for Linear systems

$$\dot{x} = Ax = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1. Let $Q = I_2$
2. Solve P from the Lyapunov equation

$$A^T P + PA = -I$$

$$\begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -2p_{11} & -4p_{12} + 4p_{11} \\ -4p_{12} + 4p_{11} & 8p_{12} - 6p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving for p_{11}, p_{12} and p_{22} gives

$$\begin{aligned} 2p_{11} &= -1 \\ -4p_{12} + 4p_{11} &= 0 \\ 8p_{12} - 6p_{22} &= -1 \end{aligned} \qquad \Longrightarrow \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 5/6 \end{bmatrix} > 0$$

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Lyapunov analysis for Linear systems

Linear system: $\dot{x} = Ax$

To check stability:

1. Find the eigenvalues of A, λ_i.
2. Verify that they are negative.

or

1. Choose an arbitrary symmetric, positive definite matrix Q.
2. Find P that satisfies Lyapunov equation

$$PA + A^T P = -Q$$

and verify that it is positive definite.

Lyapunov function: $V(x) = x^T P x$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = -x^T Q x$$