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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 5: Lyapunov stability I

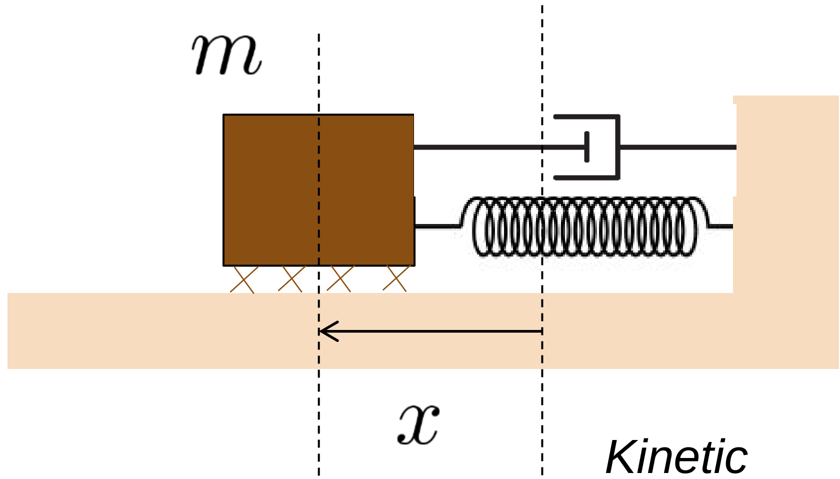
YIANNIS KARAYIANNIDIS, ASSOCIATE PROFESSOR
AUTOMATIC CONTROL, FACULTY OF ENGINEERING.

www.yiannis.info

Outline

- Physics based motivation
- Lyapunov function candidates
- Local Lyapunov Stability
- Global Lyapunov Stability
- Lyapunov stability for linear systems
- Lyapunov stability with linearization

Example (nonlinear spring with external force)



- Differential Equation

$$m\ddot{x} + \underbrace{b|\dot{x}|\dot{x}}_{\text{Nonlinear damping}} + \underbrace{k_0 x + k_1 x^3}_{\text{Hardening spring}} = 0$$

- State space representation

Position: $x_1 = x$ **Velocity:** $x_2 = \dot{x}$

- *Plug in the system dynamics*
- *Derivative along the system trajectories*

$$\boxed{m\ddot{x} = -b|\dot{x}|\dot{x} - k_0 x - k_1 x^3}$$

- Energy

$$E(x,\dot{x}) = \underbrace{\frac{m\dot{x}^2}{2}}_{\text{Kinetic}} + \underbrace{\int_0^x F_{spring}\, ds}_{\text{Potential}}$$

$$\frac{dE}{dt} = \frac{\cancel{2} m \dot{x}\ddot{x}}{\cancel{2}} + F_{spring}\dot{x}$$

$$\frac{d}{dt}E(x,\dot{x}) = \boxed{m\ddot{x}}\dot{x} + k_0 x\dot{x} + k_1 x^3\dot{x}$$

$$\int_0^x (k_0 s + k_1 s^3)\, ds = \frac{1}{2}k_0 x^2 + \frac{1}{4}k_1 x^4$$

$$\frac{d}{dt}E(x,\dot{x}) = -b|\dot{x}|\dot{x}^2 \leq 0$$

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Lyapunov function candidates

$V(x) : \Omega \rightarrow \Re$ is a $\boxed{\mathcal{C}^1 \text{ function}}$. $\frac{\partial V}{\partial x}$ exists + continuous

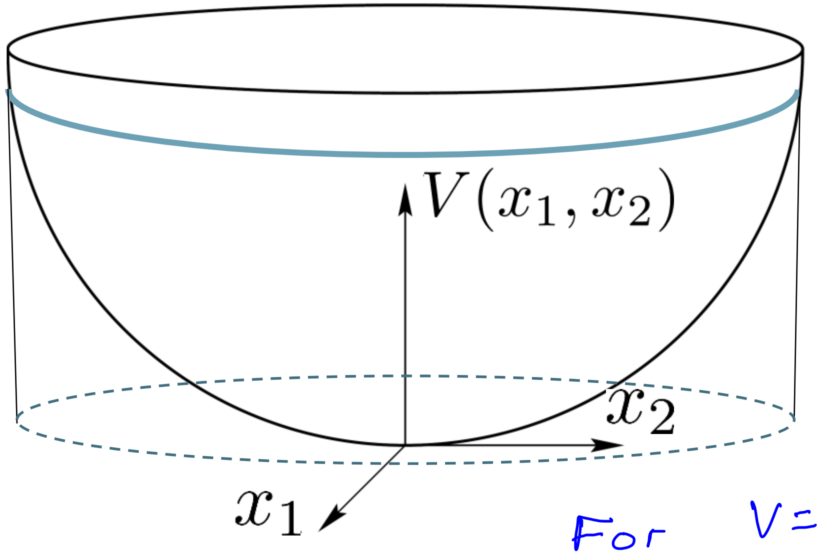
$V(x_1, x_2, \ldots)$

$V(x)$ is positive definite if $V(x) > 0 \ \forall x \neq 0$, and $V(x) = 0$ only if $x = 0$.

$V(x)$ is positive semi-definite if $V(x) \geq 0 \ \forall x \in \Omega$.

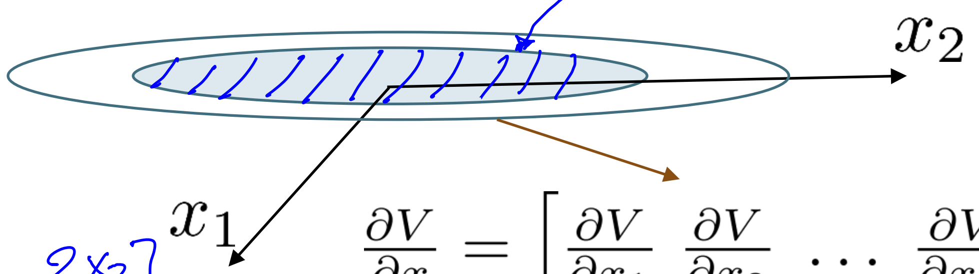
example $V(x) = x_1^2 + x_2^2$

Level sets $x_1^2 + x_2^2 = c$



Lyapunov level set where $V = c$ (c constant)

$$\Omega_c = \{x \in \Re^n : \ V(x) \leq c\}$$



$$\frac{\partial V}{\partial x} = \left[\frac{\partial V}{\partial x_1} \ \frac{\partial V}{\partial x_2} \ \ldots \ \frac{\partial V}{\partial x_n} \right]$$

For $V = x_1^2 + x_2^2$

$\frac{\partial V}{\partial x} = [2x_1 \quad 2x_2]$

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Lyapunov function candidates – positive definite functions

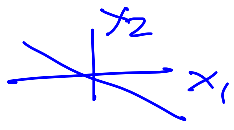
$x=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ $\quad x^*=\begin{bmatrix}0\\1\end{bmatrix}$, $x^{**}=\begin{bmatrix}0\\1\end{bmatrix}$ $\quad V(x^*)=V(x^{**})=0$

$V(x) = x_1^2$ $\rightarrow$ positive semidefinite

$V(x) = x_1^2 + a x_2^2$ if $a>0$ p.d

$V(x) = x_1^2 + (x_1 - x_2)^2$ $\quad V(x)>0 \quad V(X)=0$ when $\begin{cases} x_1=0 \\ x_2=x_1 \end{cases}$, when $x=0$

$V(x) = (x_1 + x_2)^2$ $\rightarrow$ positive semidefinite $\quad V(X)=0$ when $x_1=-x_2$



$V(x) = \int_0^{x_1} h(y)dy + x_2^2$

$V(x) = h^2(x_1) + x_2^2$

$h(y)$ *increasing function with* $h(0)=0$

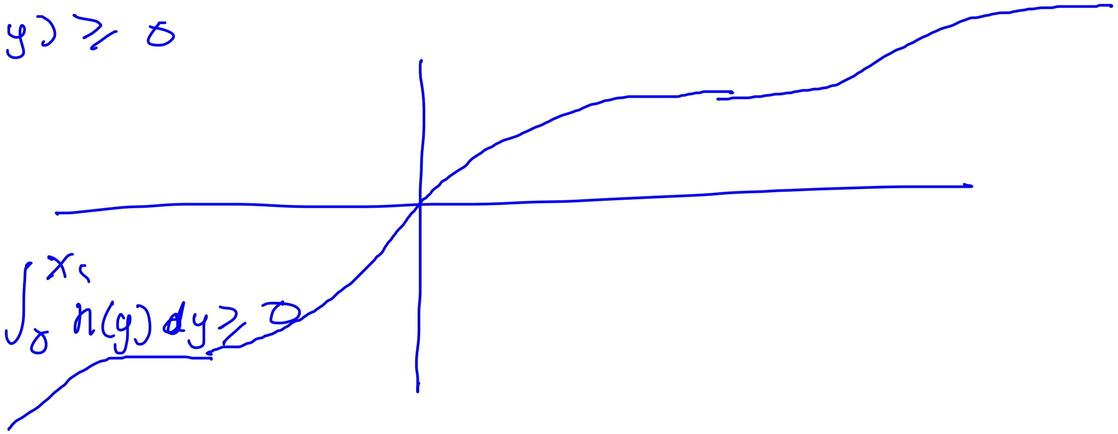
$h(y)=0$ unique solution $y^*=0$

$y\,h(y) \geq 0$

$V=\int_0^{x_1} h(y)dy + \underbrace{x_2^2}$

$\hookrightarrow$ zero for $x_2=0$

$\underset{\text{increasing}}{y>0 \Rightarrow} h(y) \geq h(0) \underset{h(0)=0}{\Rightarrow} h(y) \geq 0 \Rightarrow \int_0^{x_1} h(y)dy \geq 0$



Lyapunov function candidates – quadratic forms

For a symmetric matrix $M = M^T$ all eigenvalues are real and:

- $x^T M x > 0, \ \forall x \neq 0 \iff M$ positive definite $\iff \lambda_i(M) > 0, \ \forall i$
- $x^T M x \geq 0, \ \forall x \iff M$ positive semi-definite $\iff \lambda_i(M) \geq 0, \ \forall i$

$V = x_1^2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}^{M} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- For a symmetric matrix $M = M^T$

eigs $(M) = (0, 1)$

$\|x\|^2 = x_1^2 + \dots + x_n^2$

$$\lambda_{\min}(M)\|x\|^2 \ \leq \ x^T M x \ \leq \ \lambda_{\max}(M)\|x\|^2 \,, \quad \forall x$$

$M \rightarrow$ positive semidefinite

Proof idea: factorize $M = U \Lambda U^T$, unitary U (i.e., $||Ux|| = ||x|| \ \forall x$), $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$

$V = x_1^2 + (x_1 - x_2)^2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \overbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}^{M} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- For any matrix M

$\det(M) = 1 > 0 \mapsto$ eigs > 0

$2, 1 > 0$

$$\|Mx\| \ \leq \ \sqrt{\lambda_{\max}(M^T M)}\|x\| \,, \quad \forall x$$

$M \Rightarrow$ positive def.

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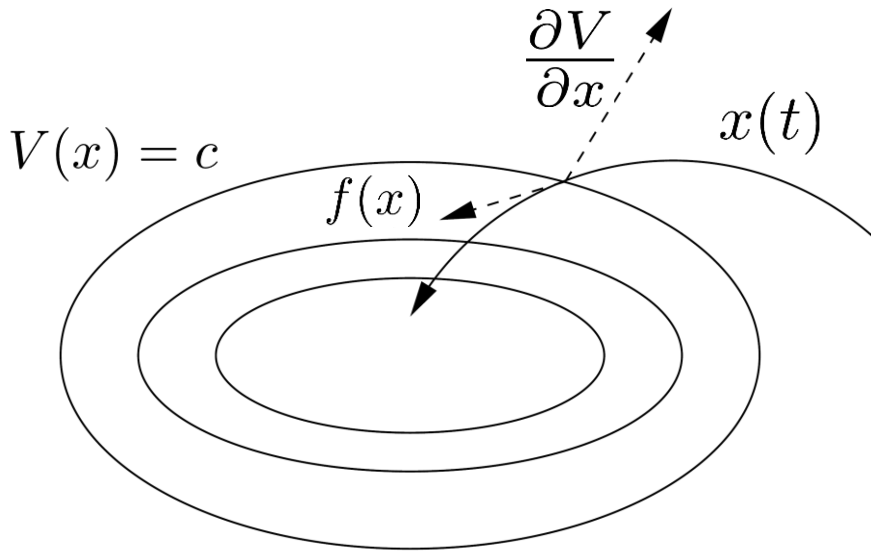
Differentiating Lyapunon function candidates along trajectories

$\dot{x} = f(x)$

$$\dot V(x) = \frac{\partial V}{\partial x}\dot x = \sum_i \frac{\partial V}{\partial x_i} f_i(x)$$

$$\frac{\partial V}{\partial x} = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots \frac{\partial V}{\partial x_n}\right]$$



Example: $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$

$$\begin{cases} \dot x_1 = & -x_1 + x_1x_2 \\ \dot x_2 = & -x_2 - x_1^2 \end{cases}$$

$$\dot V = x_1\dot x_1 + x_2\dot x_2$$
$$= -x_1^2 + \cancel{x_1^2x_2} - x_2^2 - \cancel{x_1^2x_2}$$
$$= -x_1^2 - x_2^2$$

$$\begin{cases} \dot x_1 = & -x_2 + x_1x_2 \\ \dot x_2 = & x_1 - x_1^2 \end{cases}$$

$$\dot V = [x_1 \quad x_2]\begin{bmatrix} -x_2 + x_1x_2 \\ x_1 - x_1^2 \end{bmatrix}$$
$$= \cancel{-x_1x_2} + \cancel{x_1^2x_2} + \cancel{x_2x_1} - \cancel{x_1^2x_2} = 0$$

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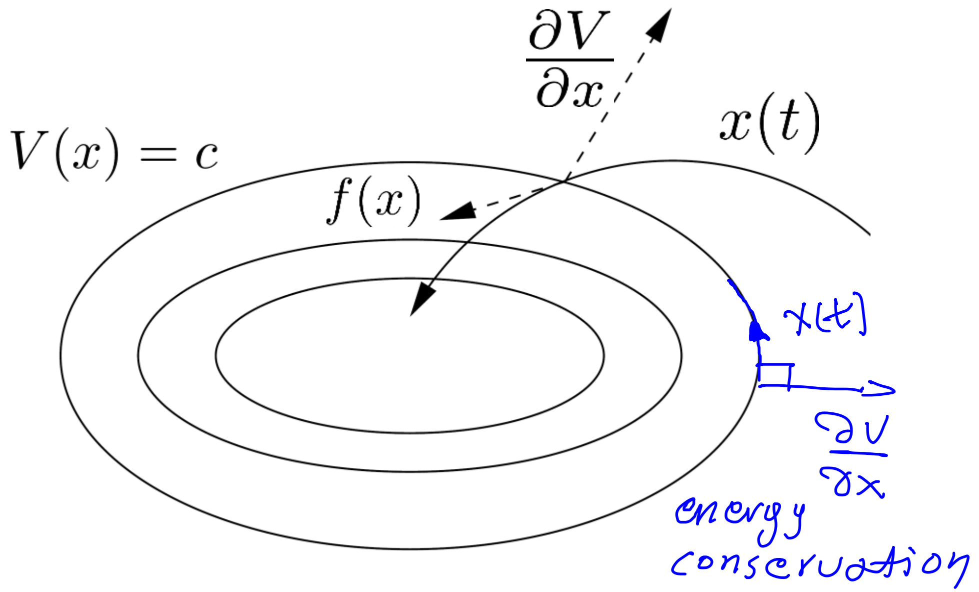
Energy conservation and dissipation

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \sum_i \frac{\partial V}{\partial x_i} f_i(x)$$



$$\frac{\partial V}{\partial x} = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots \frac{\partial V}{\partial x_n}\right]$$

[. . .]

$a^T b = a \cdot b = \|a\| \|b\| \cos\theta$

↳ angle between a, b

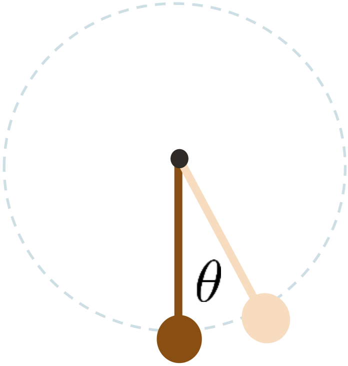
- **Energy conservation:** $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = 0$, i.e., $f(x) \perp \frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial x}$: normal to the level surface $V(x) = c$

  ↳ inner product

- **Energy dissipation:** $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0$, i.e., $f(x)$ $\frac{\partial V}{\partial x}$ forms an angle bigger than $\pi/2$ smaller than π

  (equal)

Energy conservation and dissipation (pendulum)



$$m\ell^2\ddot\theta + k\ell\dot\theta + mg\ell\sin\theta = 0$$

$$\begin{cases} \dot x_1 = & x_2 \\ \dot x_2 = & -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1, \quad b = \frac{k}{\ell} \end{cases}$$

$$V(x_1,x_2) = E(\theta,\dot\theta) = \underbrace{\frac{1}{2}m\ell^2\dot\theta^2}_{\text{kinetic}} + \underbrace{mg\ell(1-\cos\theta)}_{\text{potential}}$$

Energy

No friction $b = 0$, lossless mechanical system

Friction $b > 0$, damping

$$\dot E(\theta,\dot\theta) = m\ell^2\dot\theta\ddot\theta + mg\ell\sin\theta\,\dot\theta = \dots = -b\ell^2\dot\theta^2$$

if $b=0$ $\dot E = 0$

Local stability of an equilibrium point of a nonlinear system – Lyapunov's Direct Method

$$\dot{x} = f(x),\ f(x^\star) = 0 \qquad x^\star = 0 \in \Omega \subset \Re^n$$

$\exists\, V : \Omega \to \Re$

$V(x)$	Positive definite
$-\dot{V}(x)$	Positive semidefinite

→ **Stable Equilibrium**

$-\dot{V}(x)$ Positive definite

Stable Equilibrium + ($-\dot{V}(x)$ Positive definite) → **Asymptotically Stable Equilibrium**

$$\begin{cases} \dot{x}_1 = -x_1 + x_1 x_2 \\ \dot{x}_2 = -x_2 - x_1^2 \end{cases}, \quad V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

eq. point

$x_1 = x_2 = 0$

Stability of $(0,0)$

$$\left.\begin{array}{l} V \text{ positive definite} \\ \dot{V} = -x_1^2 - x_2^2 < 0 \quad -\dot{V} \text{ pos. def.} \end{array}\right\} \Rightarrow (0,0) \text{ asympt. stable}$$

$$\begin{cases} \dot{x}_1 = +x_2 + x_1 x_2 \\ \dot{x}_2 = -x_1 - x_1^2 \end{cases}$$

$$\left.\begin{array}{l} V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \quad \text{p.d.} \\ \dot{V} = 0 \end{array}\right\} (0,0) \text{ stable}$$

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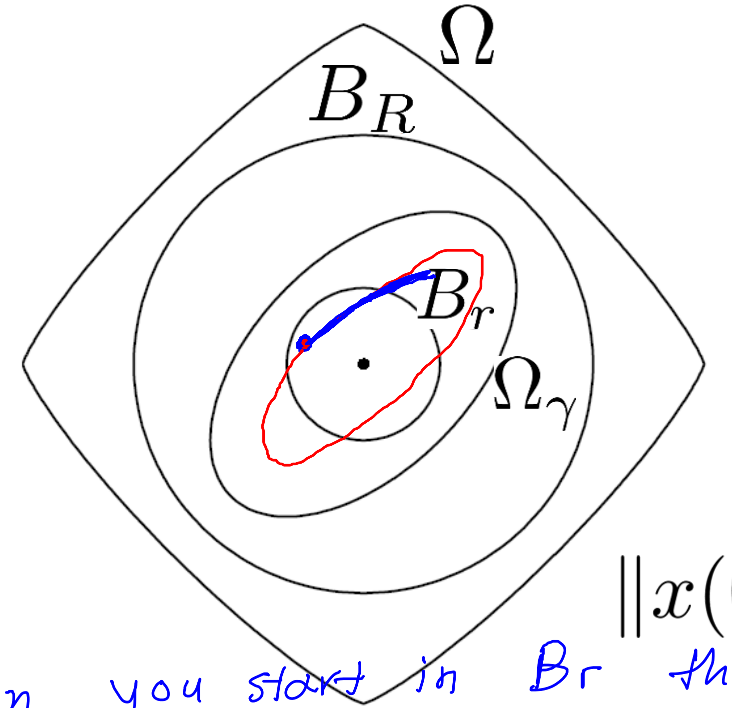
Sketchy proof of the basic Lyapunov Theorem on stability

$x^\star = 0 \qquad \forall R,\ \exists\ r(R),\ x(t)$ starts in $B_r = \{x \in \Re : \|x\| \le r\}$

remains in $B_R = \{x \in \Re : \|x\| \le R\} \subset \Omega$



Choose $\Omega_\gamma = \{x \in \Re :\ V(x) \le \gamma\} \subset B_R$

Choose $B_r = \{x \in \Re : \|x\| \le r\} \subset \Omega_\gamma$

$$\dot V(x(t)) \le 0$$

$$\|x(0)\| < r \ \Rightarrow\ V(x(t)) \le V(x(0)) := a,\ \forall t \ge 0$$

$$x(t) \in \Omega_\gamma\ ,\ \forall t \ge 0$$

$$x(t) \in B_R,\ \forall t \ge 0$$

When you start in B_r the "worst" that can hapen is to stay in $V(x) \le V(x(0)) = \alpha < \gamma$ the trajectory is in $\Omega_\gamma \subset B_R$

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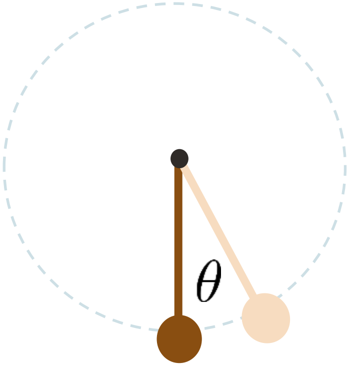
Stability analysis of eq. for the pendulum



$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1, \quad b = \frac{k}{\ell} \end{cases}$$

Equilibrium $x_2 = 0$

$\sin x_1 = 0 \Rightarrow x_1 = \pm n\pi$

By assuming $\Omega = \{|x_1| < \pi, x_2 < M\}$ $(x_1, x_2) = (0,0)$ is the only eq. in $\underline{0}$

$$V(x_1, x_2) = \frac{1}{2}m\ell^2\dot{x}_2^2 + mg\ell(1 - \cos x_1)$$ p.d in $\underline{0}$

for $|x_1| < \pi$

$(1 - \cos x_1) > 0$

$$\dot{V}(x_1, x_2) = [mg\ell \sin x_1 \;\; m\ell^2 x_2]^T \begin{bmatrix} x_2 \\ -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{bmatrix} = \ldots = -b\ell^2 x_2^2 \leq 0$$

Based on the Theorem, the conclusion we make is that the eq. $(0,0)$ is stable

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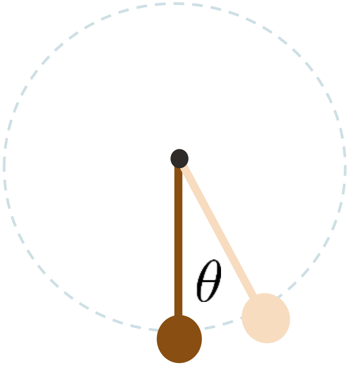
Stability analysis of eq. for the pendulum



$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1, \quad b = \frac{k}{\ell} \end{cases}$$

* We cannot make any conclusion for asymptotic stability (stability and convergence)

$$V(x_1,x_2) = \frac{1}{2}m\dot{x}_2^2 + \frac{mg}{\ell}(1-\cos x_1)$$

* we need to further investigate by e.g. considering another ~~Ly~~ Lyapunov function candidate.

$$\dot{V}(x_1,x_2) = [\frac{mg}{\ell}\sin x_1 \;\; mx_2]^T \begin{bmatrix} x_2 \\ -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1 \end{bmatrix} = \ldots = -bx_2^2$$

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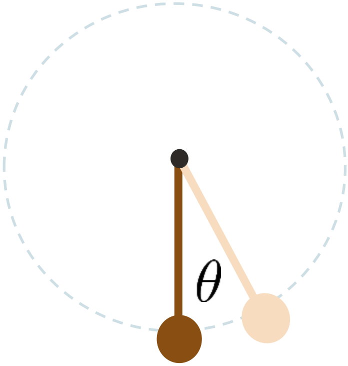
Stability analysis of eq. for the pendulum



$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & -\frac{b}{m} x_2 - \frac{g}{\ell} \sin x_1, \quad b = \frac{k}{\ell} \end{cases}$$

$$V(x_1, x_2) = \frac{1}{2} x^T P x + mg\ell(1 - \cos x_1), \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

p.d. for

$$V = \frac{1}{2} p_{11} x_1^2 + p_{12} x_1 x_2 + \frac{1}{2} p_{22} x_2^2 + mg\ell(1 - \cos x_1)$$

$p_{11} p_{22} - p_{12}^2 > 0 \quad p_{11} > 0$

$$\dot{V} = p_{11} x_1 \dot{x}_1 + p_{12} x_1 \dot{x}_2 + p_{21} x_2 \dot{x}_1 + p_{22} x_2 \dot{x}_2 + mg\ell \sin x_1 x_2$$

Differentiate V

Substitute the system dynamics

$$\dot{V} = p_{11} x_1 x_2 + p_{12} x_1 \left(-\frac{b}{m} x_2 - \frac{g}{\ell} \sin x_1\right) + p_{21} x_2^2 + p_{22} x_2 \left(-\frac{b}{m} x_2 - \frac{g}{\ell} \sin x_1\right) + mg\ell \sin x_1 x_2$$

$$\dot{V} = (p_{11} - p_{12} \frac{b}{m}) x_1 x_2 - p_{12} x_1 \frac{g}{\ell} \sin x_1 + (p_{21} - p_{22} \frac{b}{m}) x_2^2 + (mg\ell - p_{22} \frac{g}{\ell}) x_2 \sin x_1$$

group the terms that involve the states

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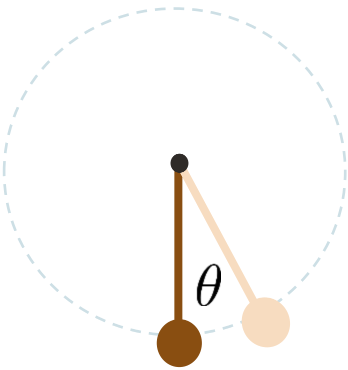
Stability analysis of eq. for the pendulum



$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & -\frac{b}{m}x_2 - \frac{g}{\ell}\sin x_1, \quad b = \frac{k}{\ell} \end{cases}$$

The terms underlined have known sign in $\underline{\Omega} = \{|x_1| < \pi, |x_2| < \infty\}$

$$V(x_1, x_2) = \frac{1}{2}x^T P x + mg\ell(1 - \cos x_1), \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$V = \frac{1}{2}p_{11}x_1^2 + p_{12}x_1x_2 + \frac{1}{2}p_{22}x_2^2 + mg\ell(1 - \cos x_1)$$

$$p_{11}p_{22} - p_{12}^2 > 0 \qquad 0 < p_{12} < b\ell^2$$

The terms in boxes are sign indefinite (should be cancelled) We don't know their sign if we don't know the functions

$$\dot{V} = (p_{11} - p_{12}\tfrac{b}{m})\boxed{x_1x_2} - p_{12}\underline{x_1}\tfrac{g}{\ell}\underline{\sin x_1} + (p_{21} - p_{22}\tfrac{b}{m})\underline{x_2^2} + (mg\ell - p_{22}\tfrac{g}{\ell})\boxed{x_2\sin x_1}$$

$x_1 \sin x_1 > 0$ in $\underline{\Omega}$

$$\boxed{p_{11} = p_{12}\frac{b}{m}} \qquad p_{12} > 0 \qquad p_{12} < p_{22}\frac{b}{m} \qquad \boxed{p_{22} = m\ell^2}$$

$$0 < p_{12} < b\ell^2$$

> 0

$$p_{11} = \frac{b^2\ell^2}{2m} \qquad p_{12} = \frac{b\ell^2}{2} \qquad p_{22} = m\ell^2$$

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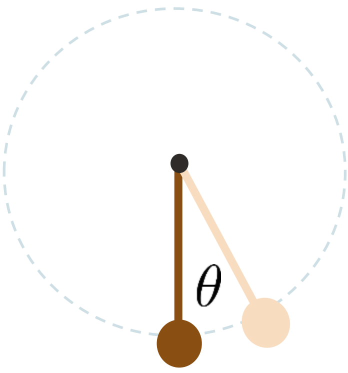
Stability analysis of eq. for the pendulum



$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & -\frac{b}{m} x_2 - \frac{g}{\ell} \sin x_1, \quad b = \frac{k}{\ell} \end{cases}$$

$$V(x_1, x_2) = \frac{1}{2} x^T P x + mg\ell(1 - \cos x_1), \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$V = \frac{1}{2} p_{11} x_1^2 + p_{12} x_1 x_2 + \frac{1}{2} p_{22} x_2^2 + mg\ell(1 - \cos x_1)$$

$$p_{11} = \frac{b^2 \ell^2}{2m} \qquad p_{12} = \frac{b\ell^2}{2}$$

$$p_{22} = m\ell^2$$

$$\dot{V} = -\frac{g\ell g}{2} x_1 \sin x_1 - \frac{b\ell^2}{2} x_2^2$$

$\left.\begin{array}{l} V \text{ pd} \\ -\dot{V} \text{ pd} \end{array}\right\} \Rightarrow (0,0)$ asymptotically stable equilibrium

Global stability of an equilibrium point of a nonlinear system – Lyapunov's Direct Method

$$\dot{x} = f(x),\; f(x^\star) = 0 \qquad x^\star = 0 \in \Re^n$$

$-\dot{V}(x)$ Positive definite

$\exists\, V : \Re^n \to \Re$

$V(x)$	Positive definite
$-\dot{V}(x)$	Positive semidefinite

→ **Stable Equilibrium**

Stable Equilibrium + $-\dot{V}(x)$ Positive definite → **Asymptotically Stable Equilibrium**

Is it enough to consider $\Omega = \Re^n$?

Study the stability of the eq. point

Example:

$$\begin{cases} \dot{x}_1 = \frac{-6x_1}{(1+x_1^2)^2} + 2x_2 \\ \dot{x}_2 = \frac{-2(x_1+x_2)}{(1+x_1^2)^2} \end{cases}$$

$$V = \frac{1}{2}\frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2$$ p. d

$$\dot{V} = \frac{x_1 \dot{x}_1}{(1+x_1^2)^2} + x_2 \dot{x}_2 = -\frac{6x_1^2}{(1+x_1^2)^4} - \frac{2x_2^2}{(1+x_1^2)^2}$$

$-\dot{V}$ pd →

asymptotically stable

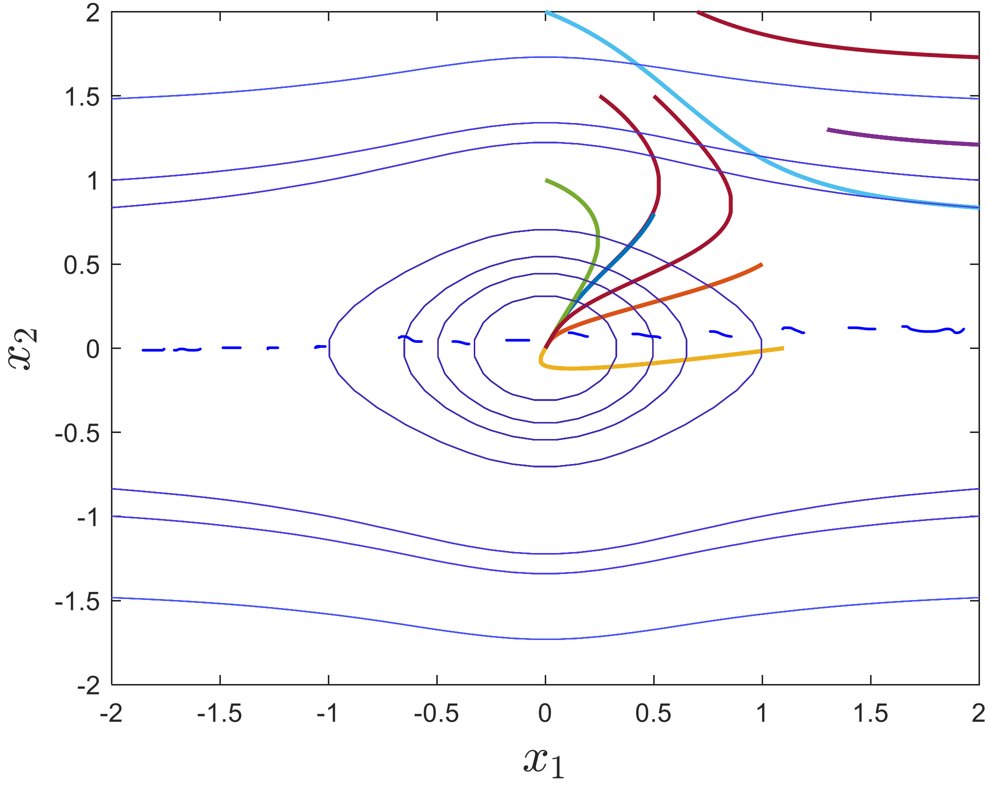
global ?

→ check the next slide

Radially unbounded functions

Example:

$$\begin{cases} \dot{x}_1 = \frac{-6x_1}{(1+x_1^2)^2} + 2x_2 \\ \dot{x}_2 = \frac{-2(x_1+x_2)}{(1+x_1^2)^2} \end{cases}$$



Radially unbounded function: $V(x) \to \infty$ as $\|x\| \to \infty$

Is $V = \frac{1}{2}\frac{x_1^2}{1+x_1^2} + \frac{1}{2}x_2^2$ radially unbounded?

No

If $\|x\| \to \infty$ e.g. $x_1 \to \infty$, $x_2 \to K$

then $V \to \frac{1}{2} + \frac{1}{2}K^2$

Find common points of $V = c$ with $x_2 = 0$ $\Rightarrow \frac{1}{2}\frac{x_1^2}{1+x_1^2} = c$

$x_1^2 - 2c x_1^2 = 2c$

$x_1^2 = \frac{2c}{1-2c}$

when $c = \frac{1}{2}$ no common points

$V = \frac{1}{2}$ * is unbounded

* closed Lyapu sets $V < \frac{1}{2}$

Lyapunov function candidates for global stability – radially unbounded functions

$$V(x) = x_1^2 + a x_2^2$$

$$V(x) = x_1^2 + (x_1 - x_2)^2$$

$$\left.\begin{array}{l} V(x) = \int_0^{x_1} h(y)dy + x_2^2 \\ \\ V(x) = h^2(x_1) + x_2^2 \end{array}\right\} \; h(y)$$

increasing function with

$$h(0) = 0$$

Lyapunov function candidates – radially functions

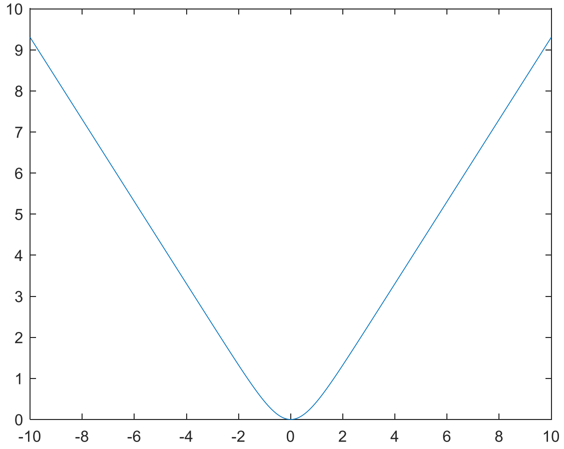
unbounded functions

$$V(x) = \int_0^{x_1} h(y)dy + x_2^2$$

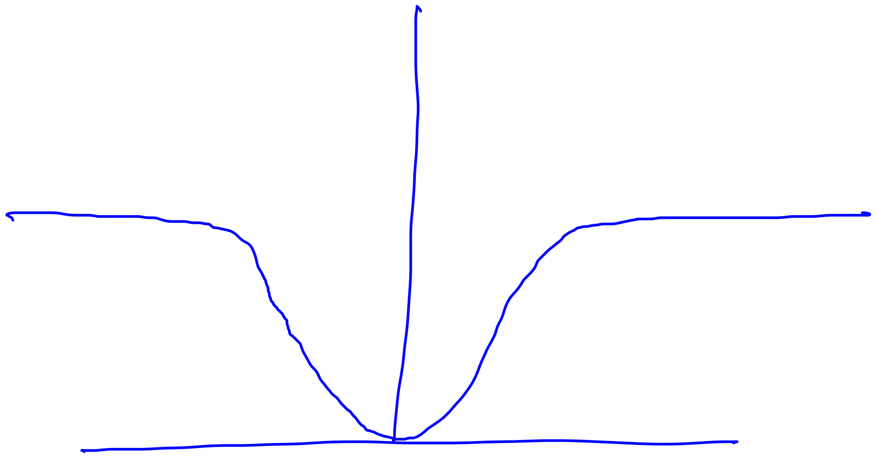
$$V(x) = h^2(x_1) + x_2^2$$

$h(y)$ *increasing function with*

$h(0) = 0$



$\boxed{\int_0^{x_1} \tanh(y)\, dy}$



$(\tanh x_1)^2$

Global stability of an equilibrium point of a nonlinear system – Lyapunov's Direct Method

$$\dot{x} = f(x),\ f(x^\star) = 0 \qquad x^\star = 0 \in \Re^n$$

$$\exists\, V : \Re^n \to \Re$$

$V(x)$	Positive definite
$V(x)$	Radially unbounded
$-\dot{V}(x)$	Positive semidefinite

→ Global

Stable Equilibrium

$-\dot{V}(x)$ Positive definite

$+$ → Global

Asymptotically Stable Equilibrium

$$\Omega = \Re^n$$

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Stability of an equilibrium point of a nonlinear system – Lyapunov's Direct Method

$$\dot{x} = f(x),\ f(x^\star) = 0 \qquad x^\star = 0 \in \Re^n$$

$$\Omega(\Re^n)$$

$$\exists\ V : \Omega(\Re^n) \to \Re$$

$V(x)$ Positive definite

$V(x)$ Radially unbounded

$-\dot{V}(x)$ Positive semidefinite

$$\Omega(\Re^n)$$

Stable Equilibrium

$-\dot{V}(x)$ Positive definite

$+$

Asymptotically Stable Equilibrium

Lyapunov stability analysis - comments

- The conditions of the Theorem are only sufficient

If conditions are not satisfied:

It does not mean that the equilibrium is unstable.

It means that the chosen Lyapunov function does not allow to make a conclusion

It requires further investigation

- ✓ *try to find another Lyapunov function*
- ✓ *Use other Theorems ☺*

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Lyapunov analysis for Linear systems

Linear system: $\dot{x} = Ax$

$$\dot{x} = Ax \quad = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To check stability:

1. Find the eigenvalues of A, λ_i.
2. Verify that they are negative.

Eigenvalues of A : $\{-1, -3\}$

$\Rightarrow$ (global) asymptotic stability.

Try to prove stability with:

$$V(x) = \|x\|^2 = x^T x = x_1^2 + x_2^2$$

$$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2$$

$$= -x_1^2 + 4x_1x_2 - 3x_2^2$$

$$= -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\det(M) = 3 - 4 = -1$

$-\dot{V}$ is not p.d.

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