LUND UNIVERSITY

FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 4: Describing function analysis

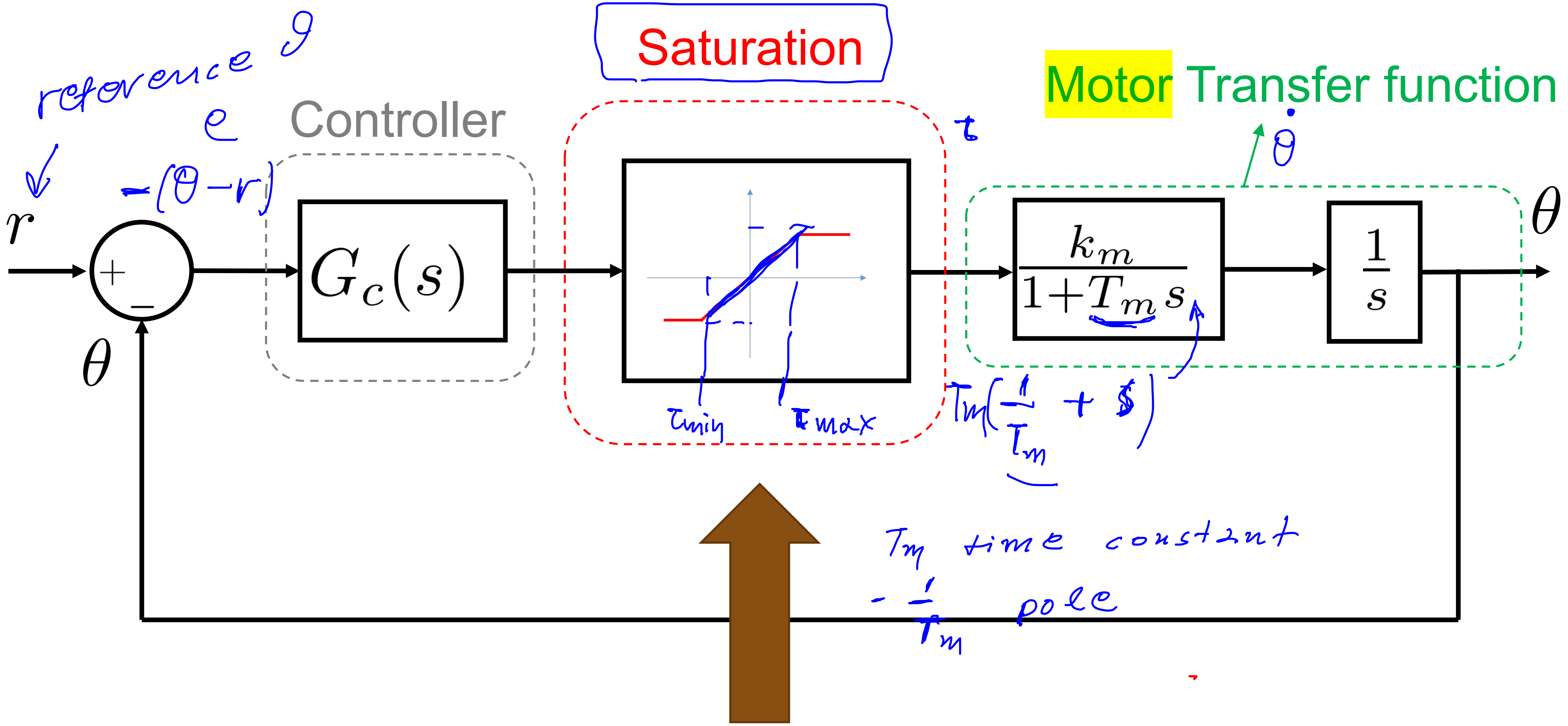
YIANNIS KARAYIANNIDIS, ASSOCIATE PROFESSOR
AUTOMATIC CONTROL, FACULTY OF ENGINEERING.

www.yiannis.info
yiannis@control.lth.se

Outline

- How to obtain a describing function for a nonlinear element in an “almost” linear system
- Prediction of oscillations based on extended Nyquist Criterion and the describing function of the nonlinearity

Motivation: Nonlinearities in the control system



Nonlinear element

Motivation: Nonlinearities

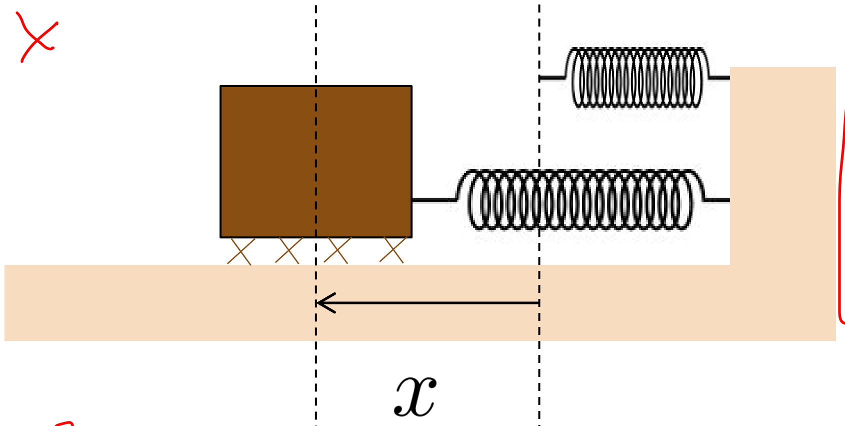
- The physical system (the plant) may contain nonlinearities

$$m\ddot{x} + d\dot{x} + kx + \underbrace{k_h x^3}_{u} = 0 \Rightarrow$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{1}{m}\left(kx_1 + k_h x_1^3 + d x_2\right) \end{cases}$$

$y = Cx$

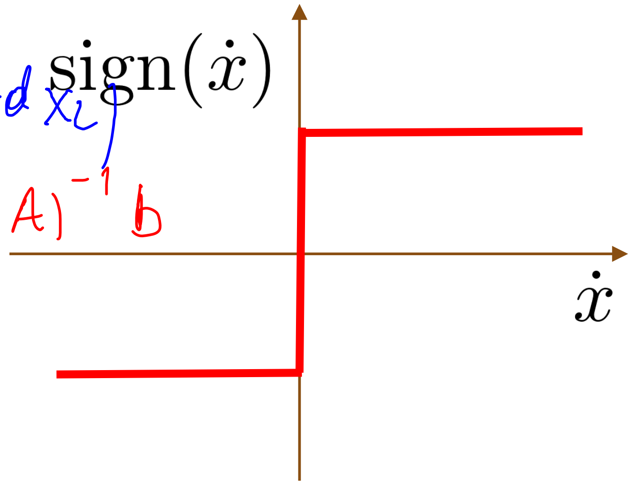
$H(s) = C(sI - A)^{-1}B$

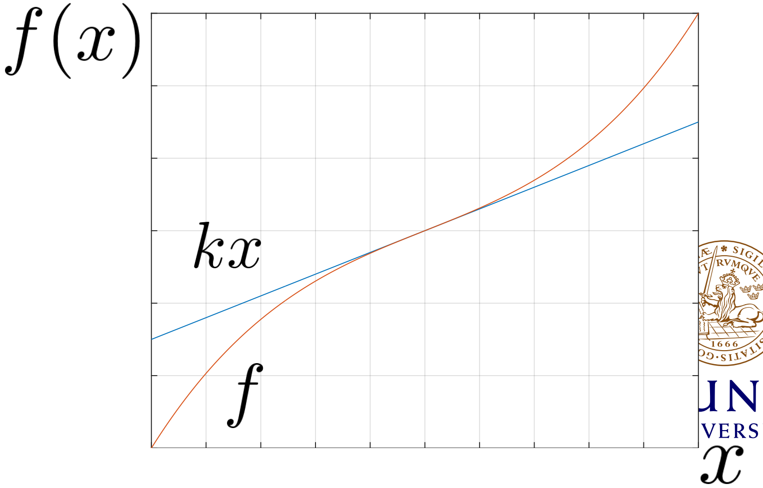


$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{1}{m}\left(kx_1 + d x_2\right) \end{cases} - \frac{1}{m}u$$

$u = k_h x_1^3$

LT: $ms^2X(s) + dsX(s) + kX(s) = -U(s)$



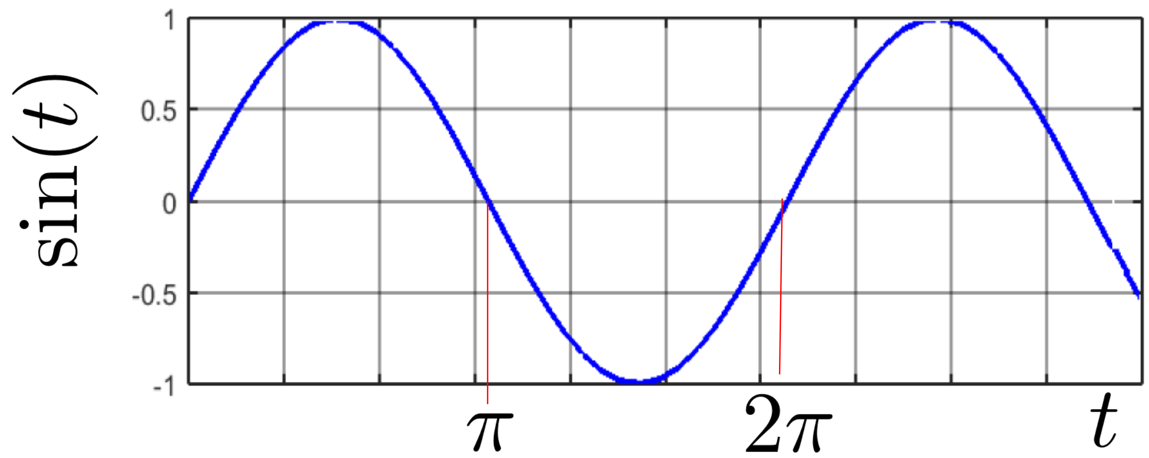


Coulomb Friction $\quad -k_c\,\text{sign}(\dot{x})$

Hardening spring $\quad f = kx + k_h x^3$

Motivation: prediction of persistent oscillations (limit cycles)

- Oscillations can be desirable: electronic oscillators used in laboratories.

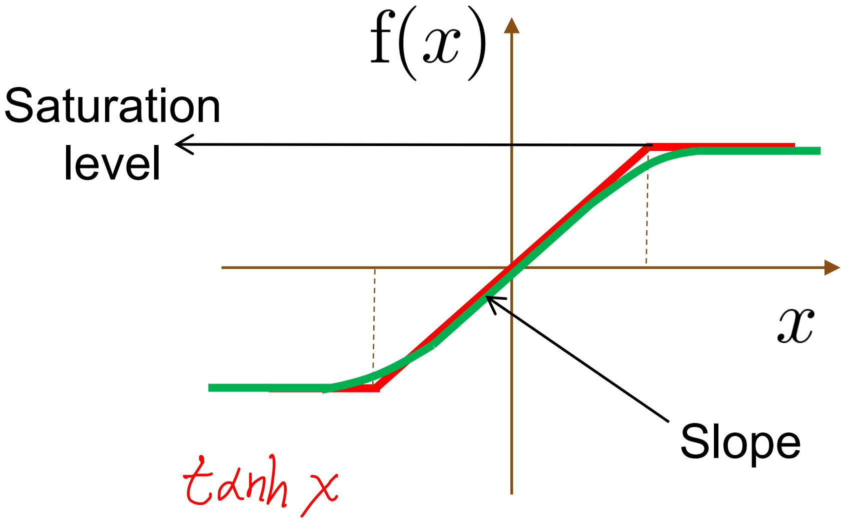


- Oscillations are undesirable
  - Oscillations are a sign of instability, tend to cause poor control accuracy
  - Constant oscillations can increase wear or even cause mechanical failure

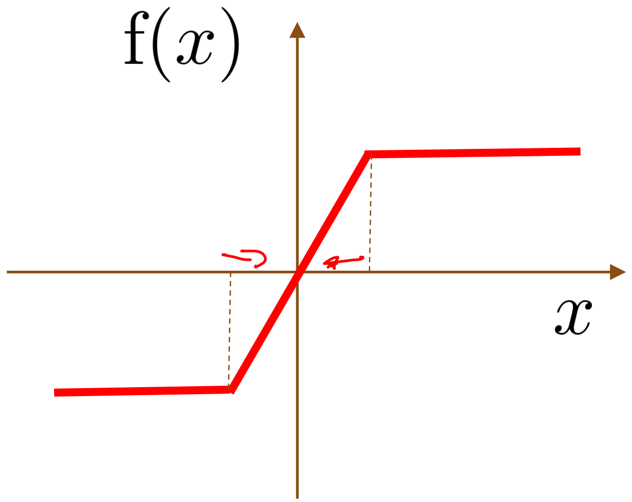
LUND UNIVERSITY

Nonlinearities: Single-valued nonlinearities

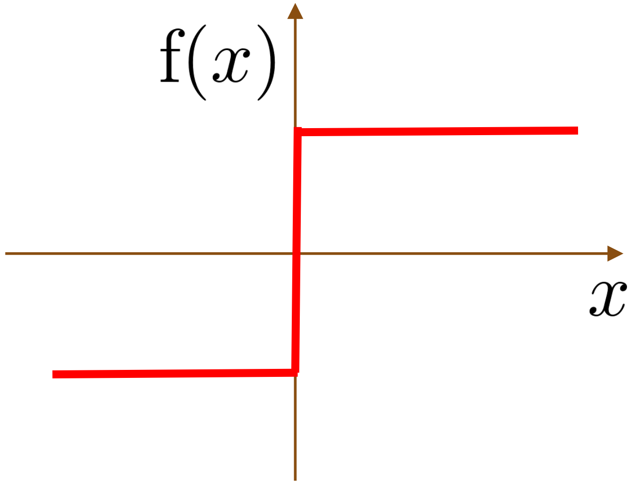
- **Saturation nonlinearity**



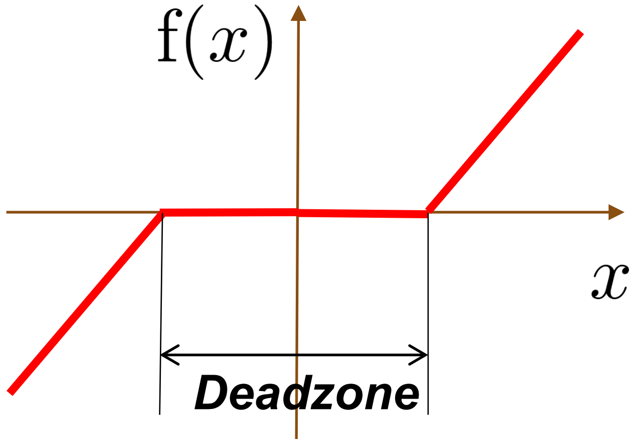
Increasing slope



- **On-Off (relay) nonlinearit**

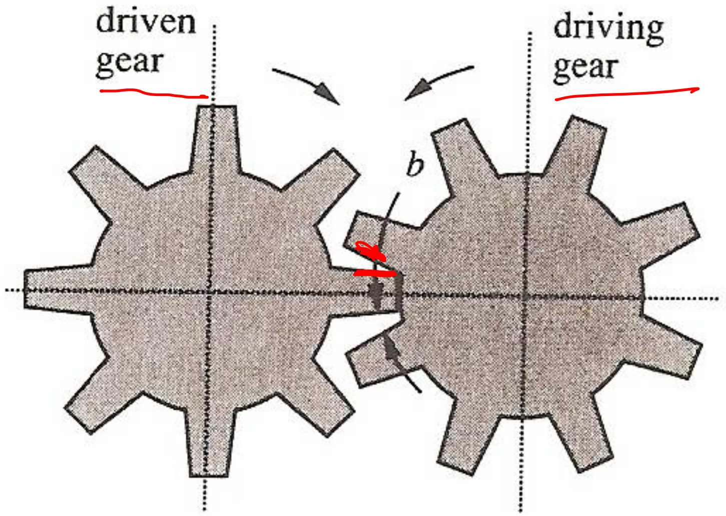


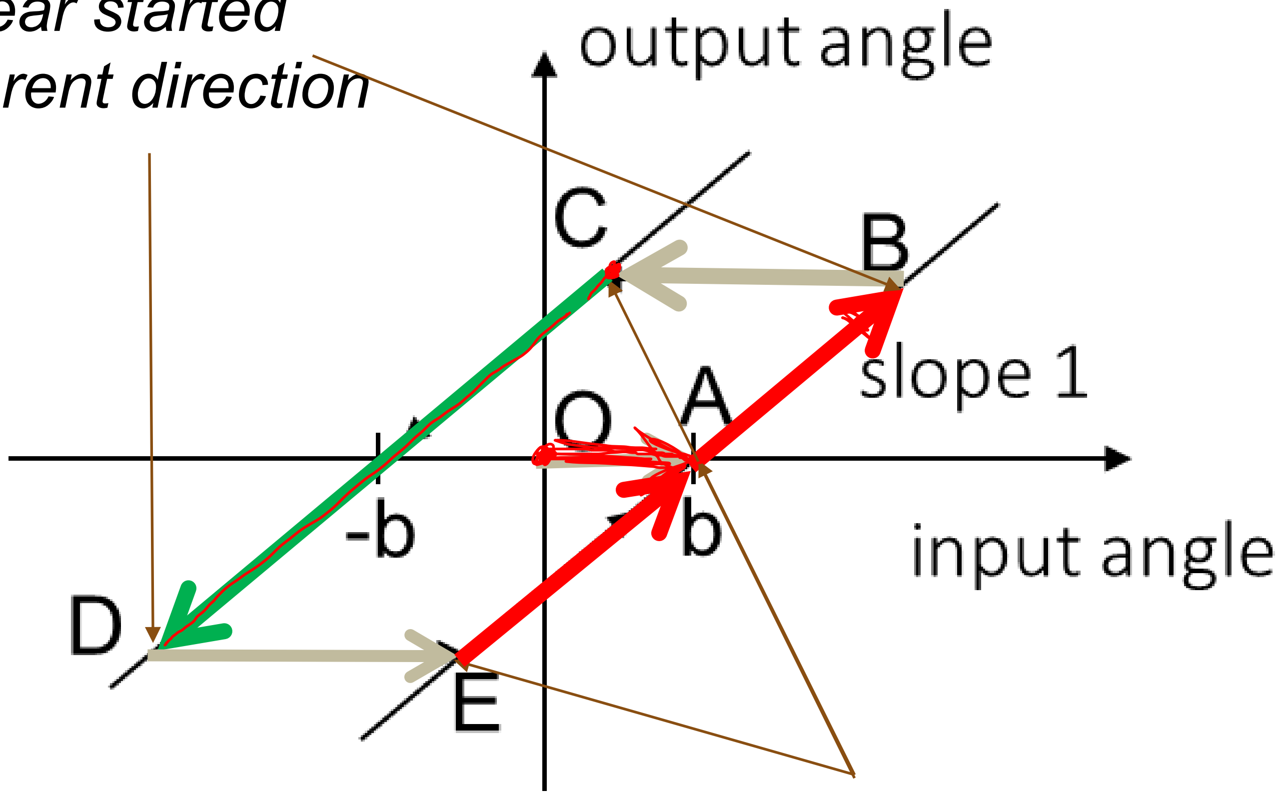
- **Deadzone nonlinearity**



Nonlinearities: Backlash

The input gear started rotating to different direction





- Multi-valued
- The output depends on the input and the history of the input

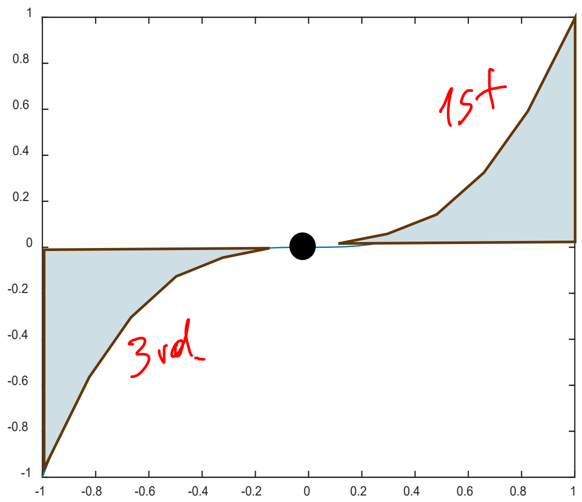
The output gear does not move until contact is (re)established

Contact is achieved

Odd and even functions

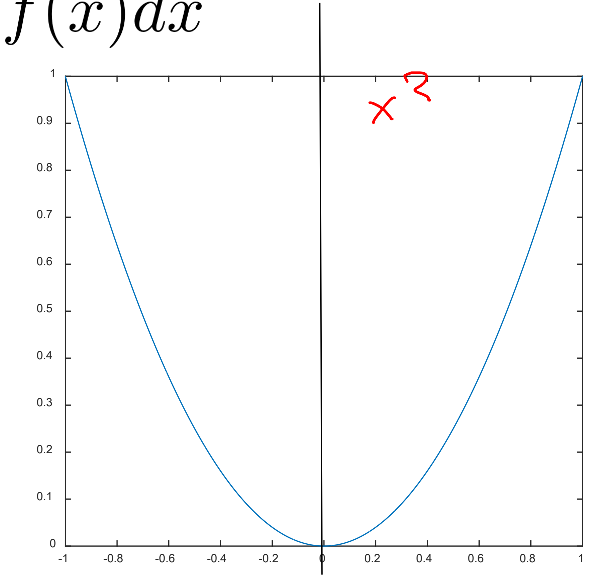
- Odd function $f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)dx = 0$$



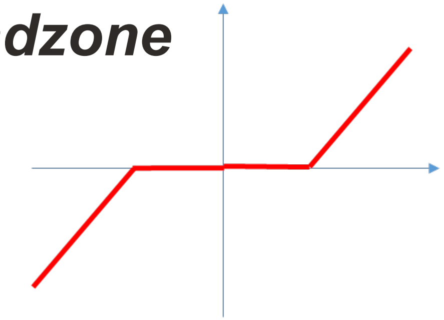
- Even function $f(-x) = f(x)$

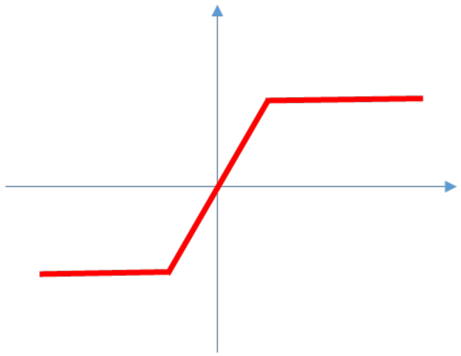
$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$



- *Examples*

Deadzone





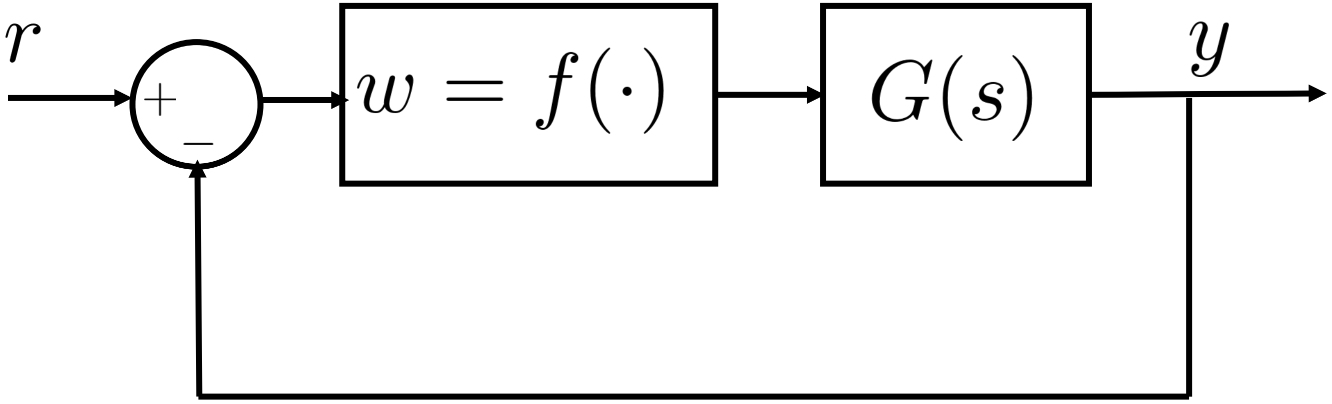
Saturation

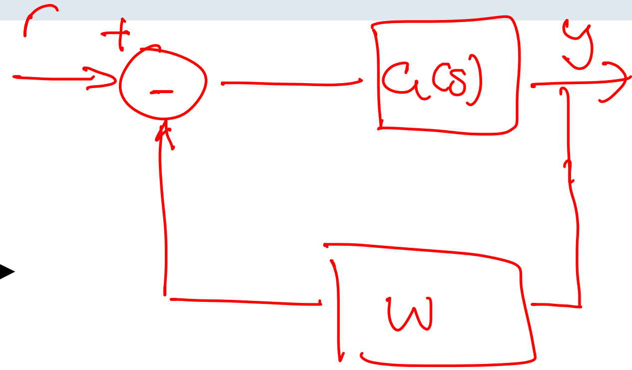
Odd functions with

$$xf(x) \geq 0$$

LUND UNIVERSITY

Describing function analysis





- Assumptions

  Single, odd, time-invariant nonlinear element $f(\cdot)$

  Low-pass transfer function $G(s)$

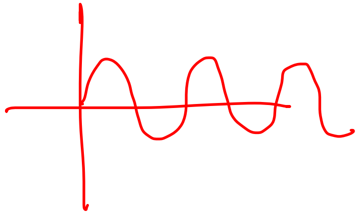
- Replace the nonlinearity with a quasi-linear component

- Use tools from linear control systems design to examine the existence of oscillations

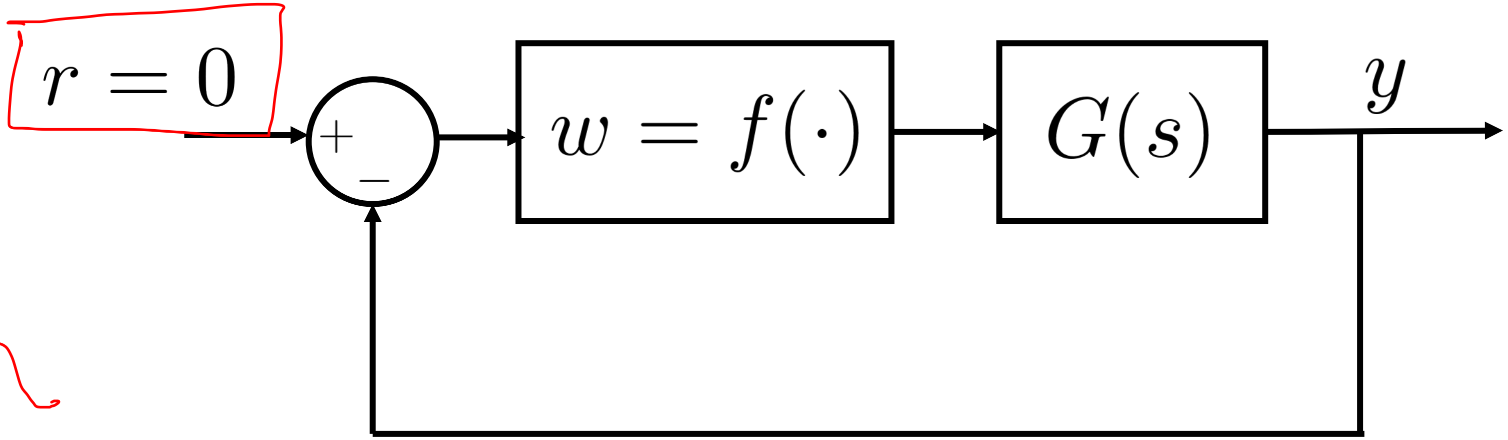
Describing function analysis

- Form of the nonlinear system

r=0



r=1

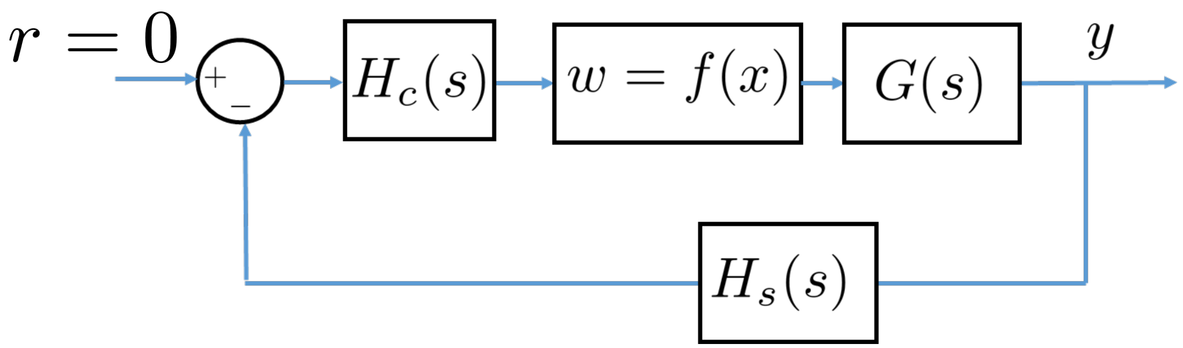


- Reference is set zero to study self-sustained oscillations

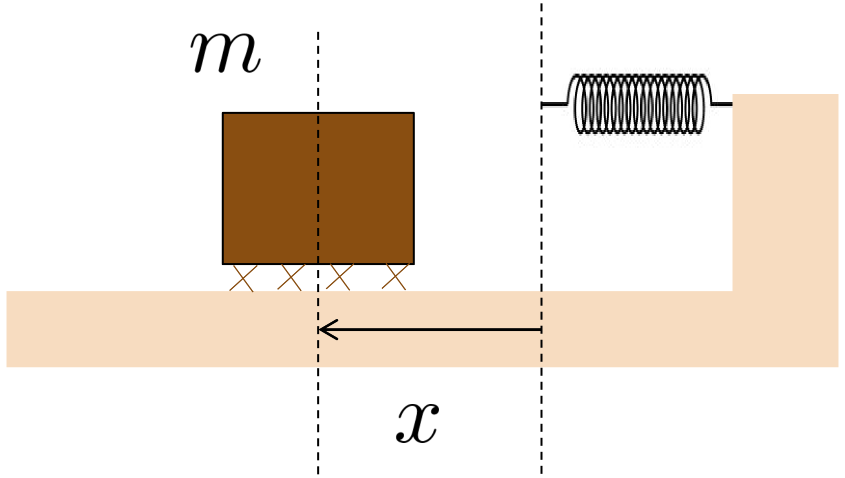
- “almost” linear system or genuinely nonlinear system (written as shown in the block diagram)

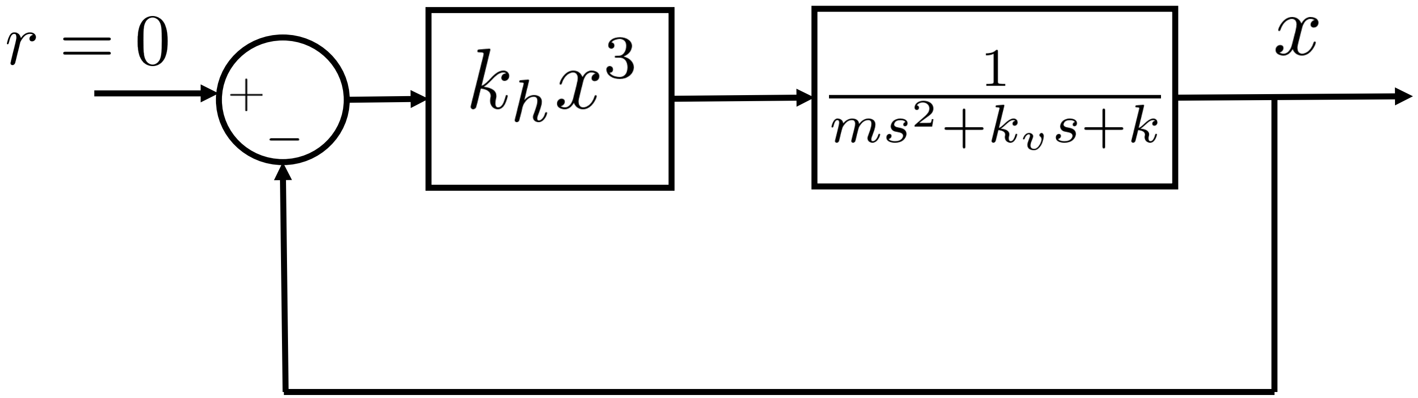
“Almost” linear systems

- Linear Control Design and linear system
- Implementation involves hard nonlinearities, e.g. actuator saturation or sensor dead-zones
- Contain hard nonlinearities in the control loop but are otherwise linear



Quiz: Write the nonlinear system in a feedback form where the nonlinearity is in a block





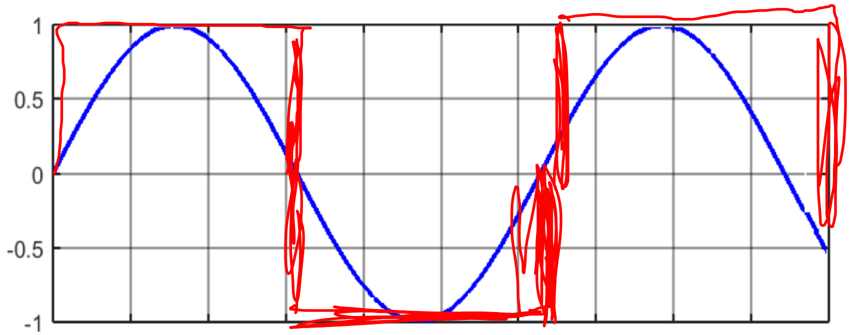
Viscous Friction $-k_v \dot{x}$

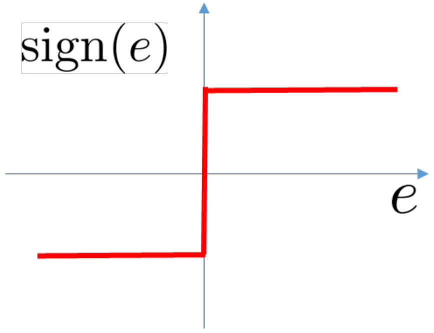
Hardening spring $f = -kx - k_h x^3$

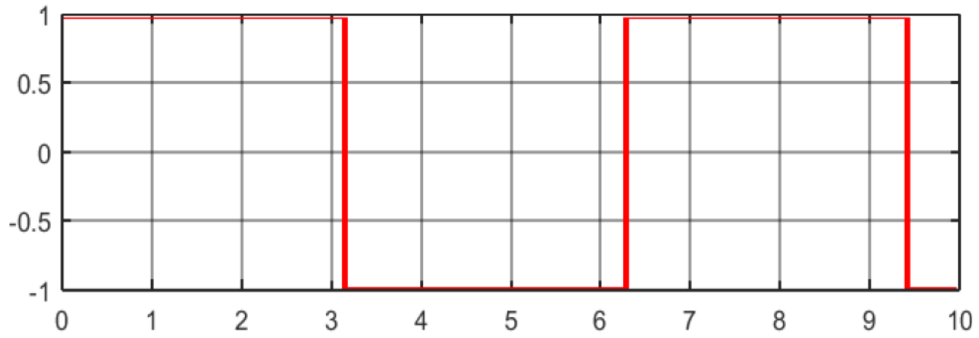
Fourier Transformation

Input $e(t) = A\sin(\omega t)$

Output $w(t) = f(e) = f(A\sin(\omega t))$







Output – Periodic function $w(t+T) = w(t)$

Fourier Transformation $w(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)d(\omega t)$

0 for odd w

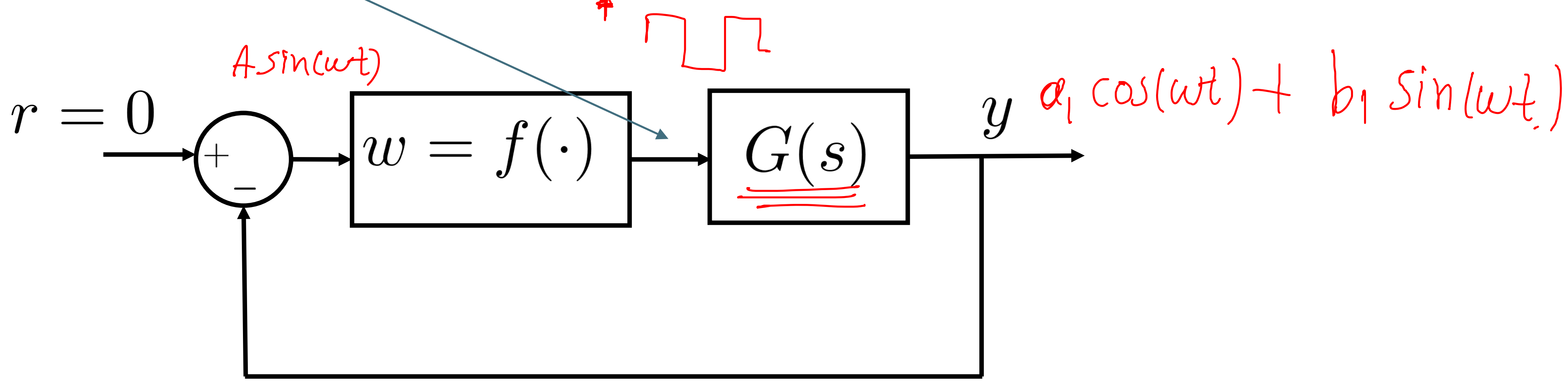
$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\cos(n\omega t)d(\omega t)$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\sin(n\omega t)d(\omega t)$

LUND UNIVERSITY

The linear transfer function as a low-pass filter

$$w(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$



- If the transfer function is acting as a low pass filter the output y will be mainly affected by the first harmonic of w

$$w(t) = w_1(t) = a_1 \cos(\omega t) + b_1 \sin(\omega t)$$

- The method is based on approximations (heuristic)

LUND UNIVERSITY

The linear transfer function as a low-pass filter

- *"Filtering" Assumption: the first harmonic is taken as output of the nonlinear block*

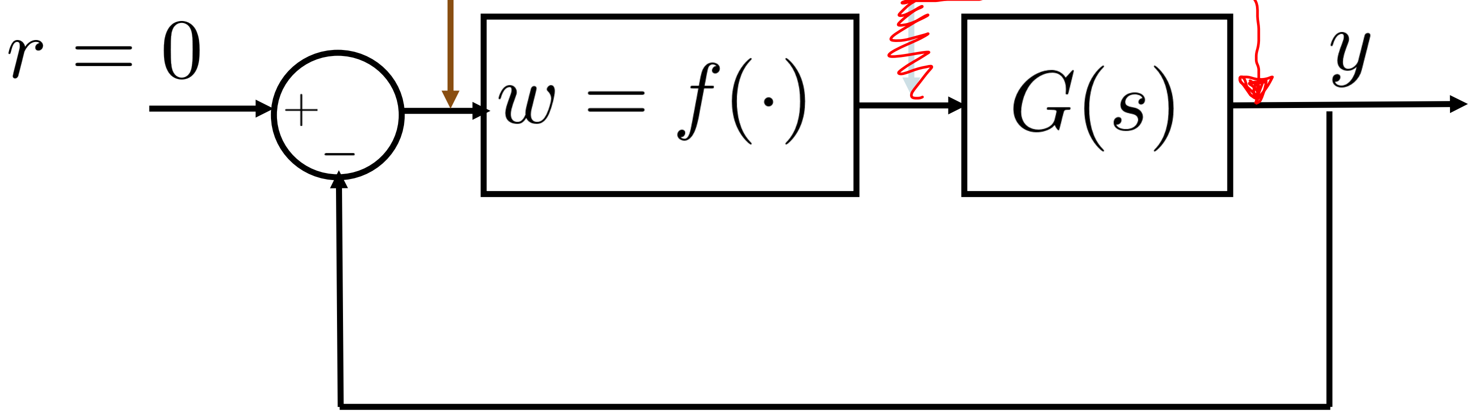
$$A\sin(\omega t)$$

$$w_1(t) = a_1\cos(\omega t) + b_1\sin(\omega t)$$

$$w_1(t) = M\sin(\omega t + \phi)$$

$$M = \sqrt{a_1^2 + b_1^2}$$

$$\phi = \arctan(a_1, b_1)$$



Describing Function

Input of the nonlinear element

$$Ae^{j\omega t}$$

$$\rightarrow \boxed{w = f(\cdot)} \rightarrow$$

Output of the nonlinear element

$$w_1(t) = Me^{(j\omega t + \phi)}$$

$$w_1(t) = (b_1 + ja_1)e^{j\omega t}$$

- Describing function definition

$$N(A, \omega) = \frac{\text{Output}}{\text{Input}} \Rightarrow N(A, \omega) = \frac{Me^{(j\omega t + j\phi)}}{Ae^{j\omega t}} = \frac{M \cancel{e^{j\omega t}} e^{j\phi}}{A \cancel{e^{j\omega t}}}$$

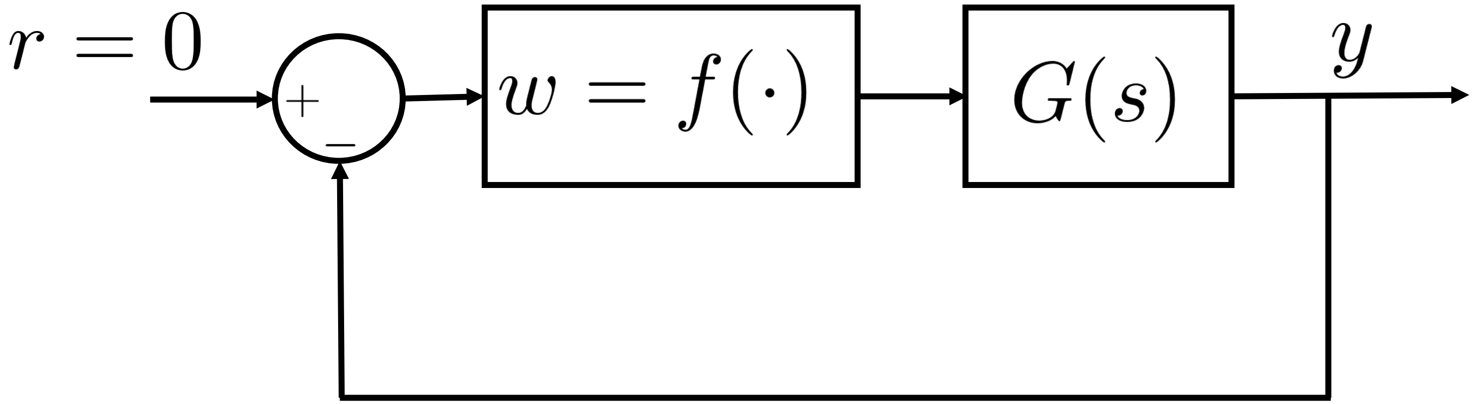
$$\Rightarrow N(A, \omega) = \boxed{\frac{M}{A}e^{j\phi}}$$

LUND UNIVERSITY

Describing Function (cont.)

$$N(A,\omega) = \frac{M}{A} e^{j\phi}$$

$$N(A,\omega) = \frac{1}{A}(b_1 + ja_1)$$

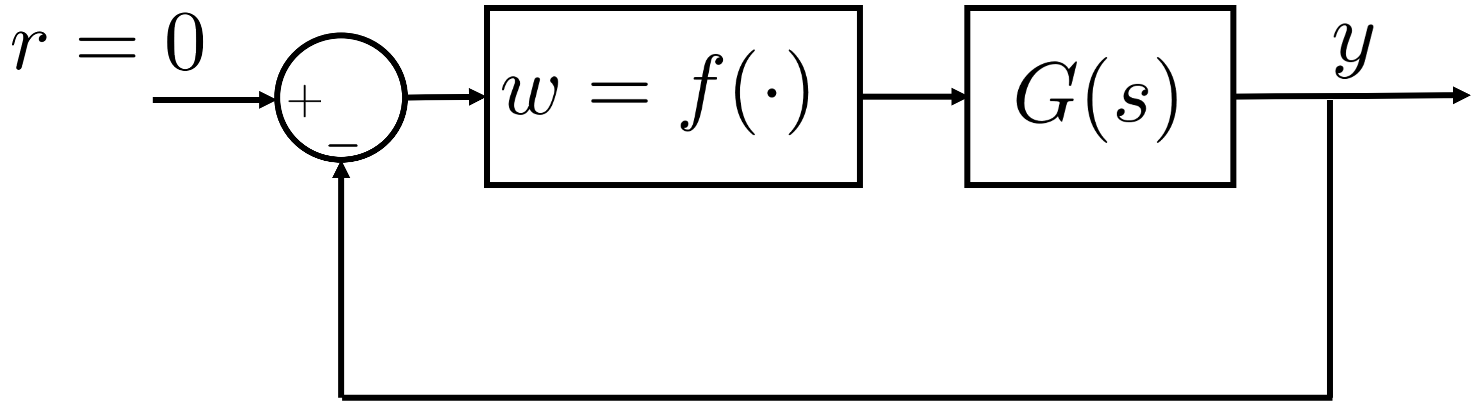


- Extension of the notion of frequency response for systems with nonlinearities
- Depends on the amplitude of the input signal in contrast to the frequency response for linear systems

Describing Function –special cases

$$N(A,\omega) = \frac{M}{A} e^{j\phi}$$

$$N(A,\omega) = \frac{1}{A}(b_1 + j a_1)$$



- It is **real** and **independent of the frequency** when the non-linearity is **single-valued**
- Why? $a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\cos(\omega t)d(\omega t) \longrightarrow$ • Imaginary part

  $b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\sin(\omega t)d(\omega t) \longrightarrow$ • Real part

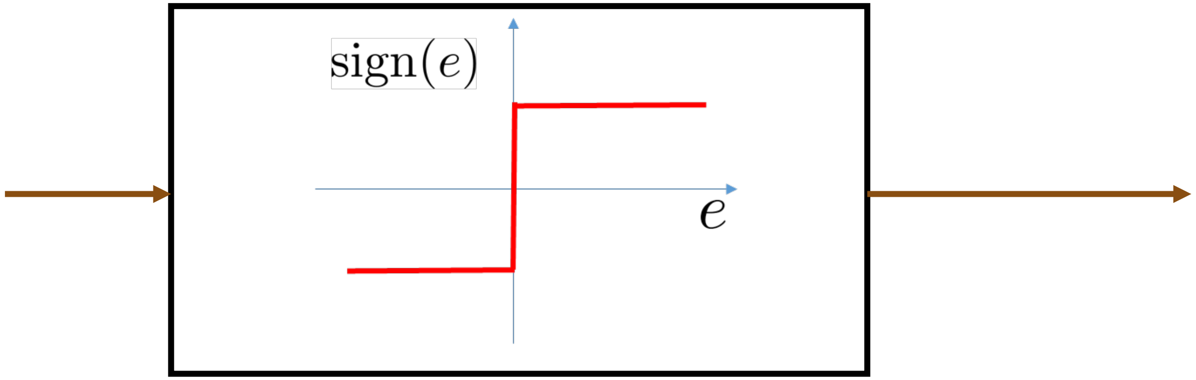
LUND UNIVERSITY

Describing Function – Example

$$N(A,\omega) = \frac{1}{A}(b_1 + j a_1)$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\cos(\omega t)d(\omega t)$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\sin(\omega t)d(\omega t)$$



Describing Function – Example

$w(t) = \text{sign}(\sin \omega t)$

$$N(A,\omega) = \frac{1}{A}(b_1 + ja_1)$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\cos(\omega t)d(\omega t)$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\sin(\omega t)d(\omega t)$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \text{sign}[\sin(\omega t)]\cos(\omega t)d(\omega t) = 0$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\sigma d\sigma = -\frac{1}{\pi}\int_{-\pi}^{0} \cos\sigma d\sigma + \frac{1}{\pi}\int_{0}^{\pi} \cos\sigma d\sigma = 0$$

$N(A,\omega) = \frac{b_1}{A} = \frac{4}{A\pi}$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \text{sign}[\sin(\omega t)]\sin(\omega t)d(\omega t)$$

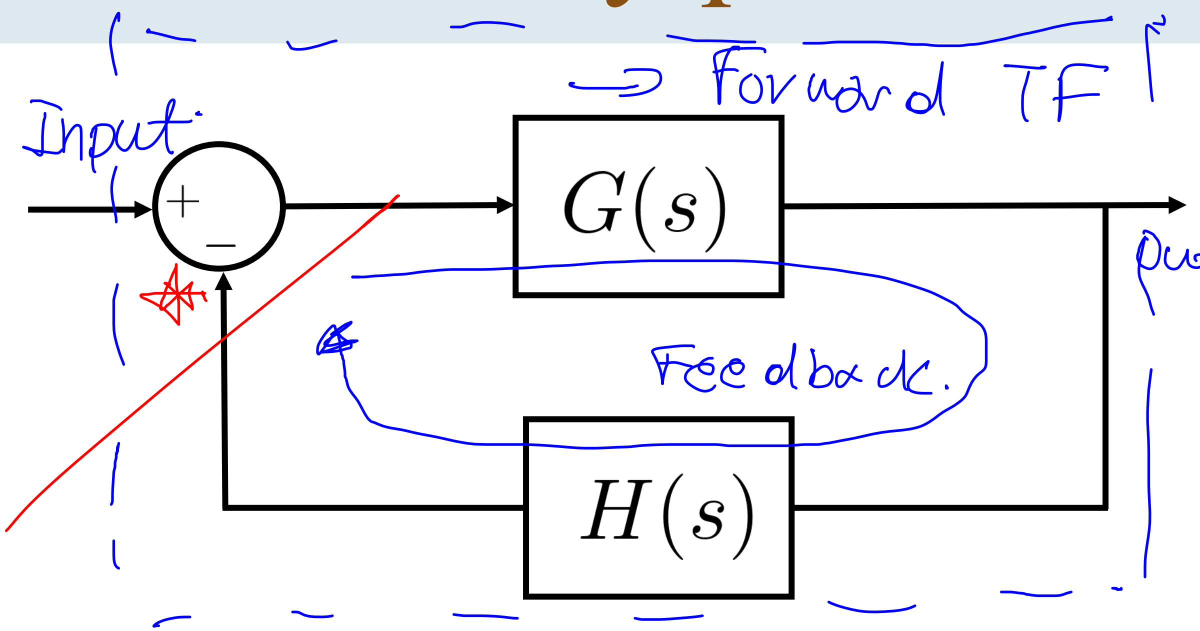
$$\boxed{N(A,\omega) = \frac{4}{A\pi}}$$

$$b_1 = \frac{2}{\pi}\int_{0}^{\pi} \sin\sigma d\sigma = \frac{2}{\pi}[-\cos\sigma]_0^{\pi} = \boxed{\frac{4}{\pi}}$$

LUND UNIVERSITY

Nyquist criterion: Definitions

$$1 + \frac{s+1}{(s-1)(s-2)} = \frac{(s-1)(s-2) + (s+1)}{(s-1)(s-2)}$$



- The **characteristic equation** of the system:

$$\Delta(s) = 1 + G(s)H(s) = 0$$

- Poles of $\Delta(s) \rightarrow$ poles the OLS system
- Zeros of $\Delta(s) \rightarrow$ poles of the CLS system

- Closed loop Transfer Function

$$\frac{G(s)}{1+G(s)H(s)}$$

- Open loop Transfer Function

$$G(s)H(s) = \frac{a_m s^m + \cdots + a_1 s + a_0}{b_n s^n + \cdots + b_1 s + b_0}$$

$m \leq n$ for proper/strictly proper transfer function

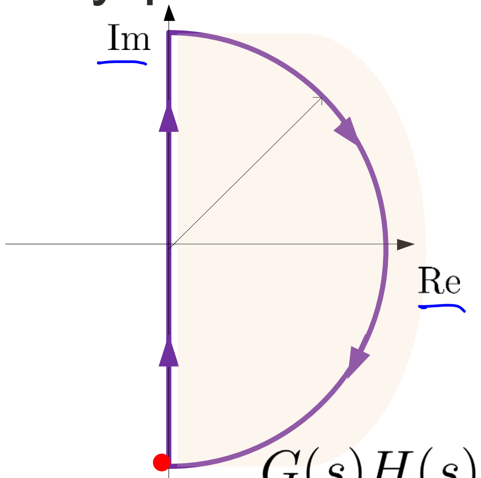
- Example:

✔ $$G(s)H(s) = \frac{s+1}{(s-1)(s-2)}$$

LUND UNIVERSITY

Nyquist contour and plot

- Nyquist contour

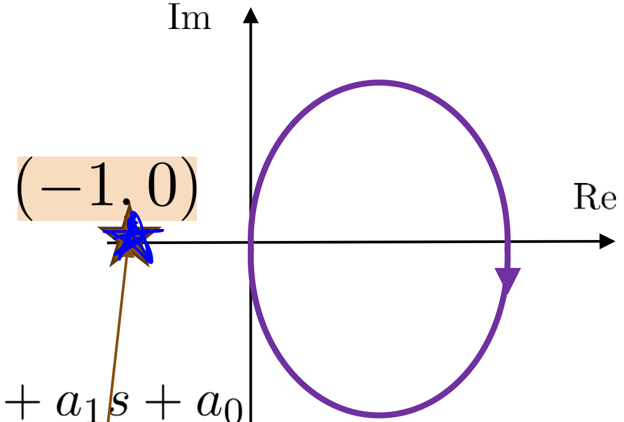


H(s) G(s)

$$G(s)H(s) = \frac{a_m s^m + \cdots + a_1 s + a_0}{b_n s^n + \cdots + b_1 s + b_0}$$

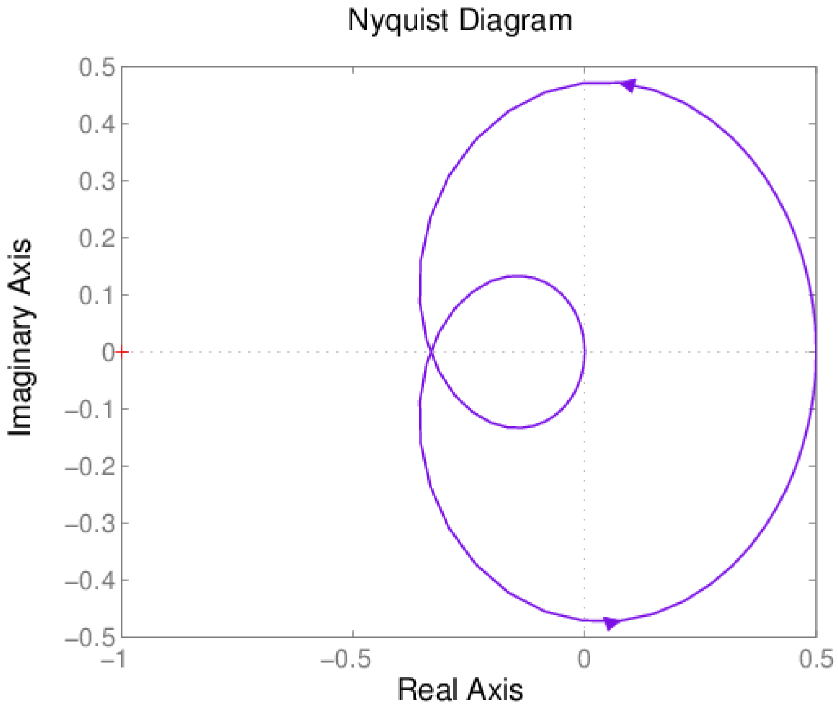
A path that encircles the right-half s plane

- Nyquist plot



Number of clockwise encirclements of the point (-1,0)

Example



$$G(s)H(s) = \frac{s+1}{(s-1)(s-2)}$$

$G(j\omega)H(j\omega) = R(\omega) + jIm(\omega)$

- Nyquist Criterion

$$P_{CL} - P_{OL} = \text{N(-1,0)}$$

LUND UNIVERSITY

Nyquist Criterion

- *The number of unstable Closed Loop Poles is equal to the number of open loop poles with positive real part plus the number of clockwise encirclements of the point (-1,0)*

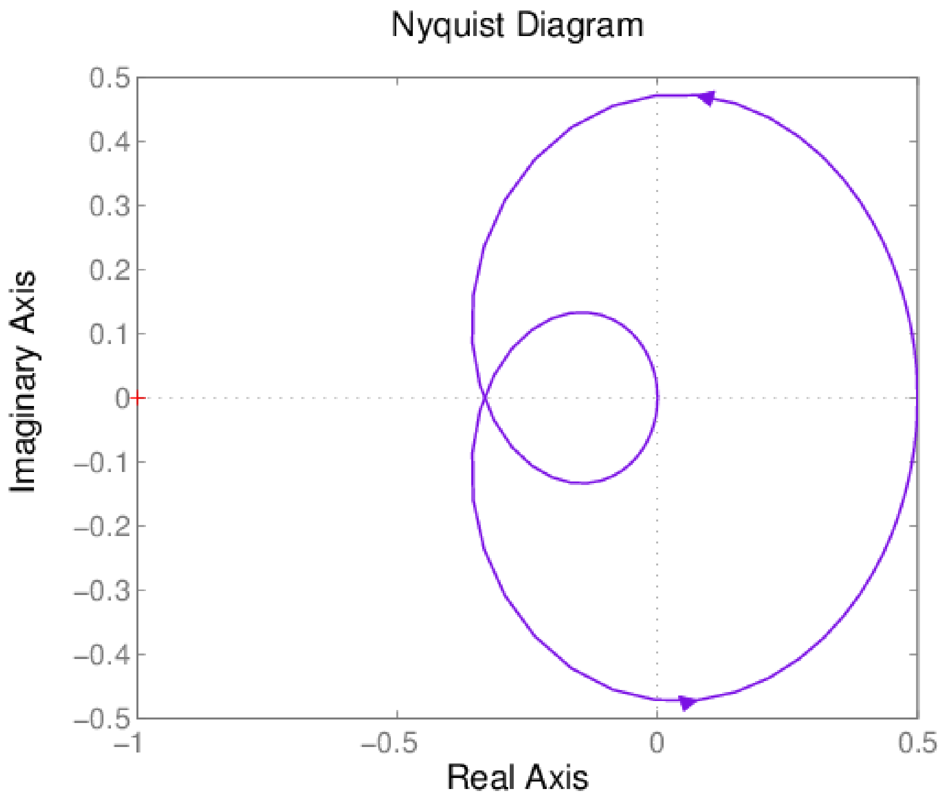
$$P_{CL} = \mathrm{N}(-1, 0) + P_{OL}$$

- *Given a stable open loop system, the closed loop is stable if the Nyquist plot of the open loop system does not encircle the point (-1,0).*

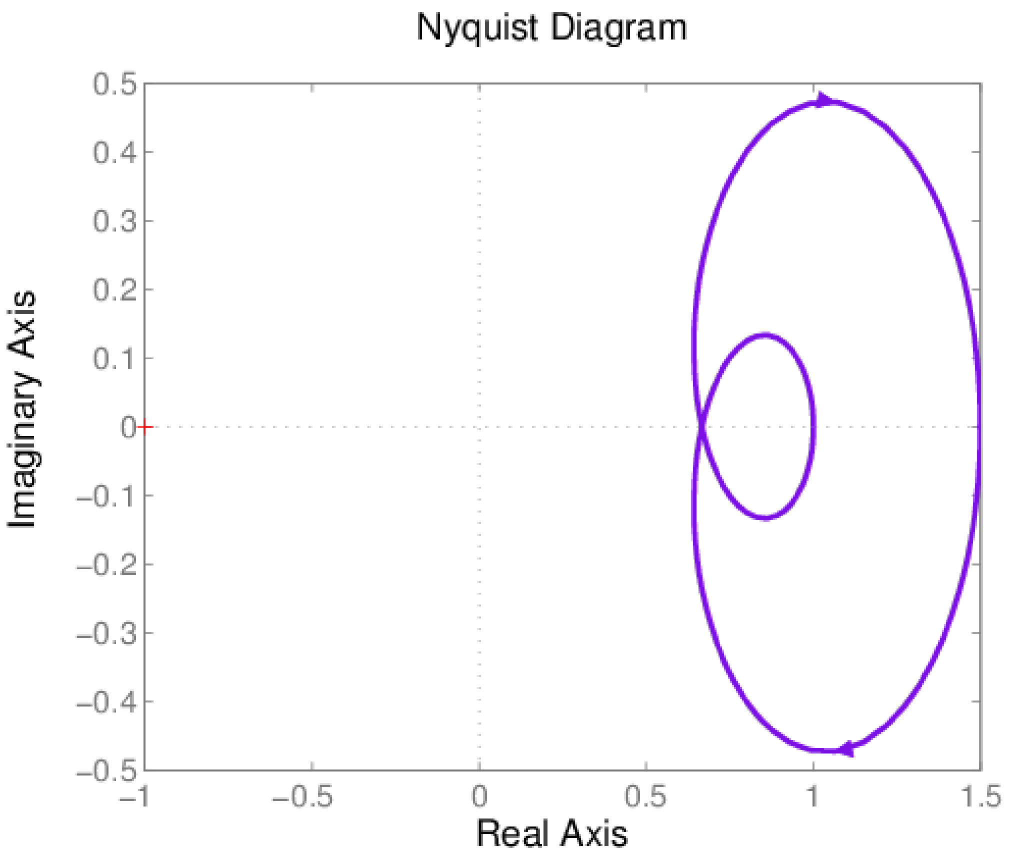
LUND UNIVERSITY

Nyquist Criterion: Quiz

$$G(s)H(s) = \frac{s+1}{(s-1)(s-2)}$$

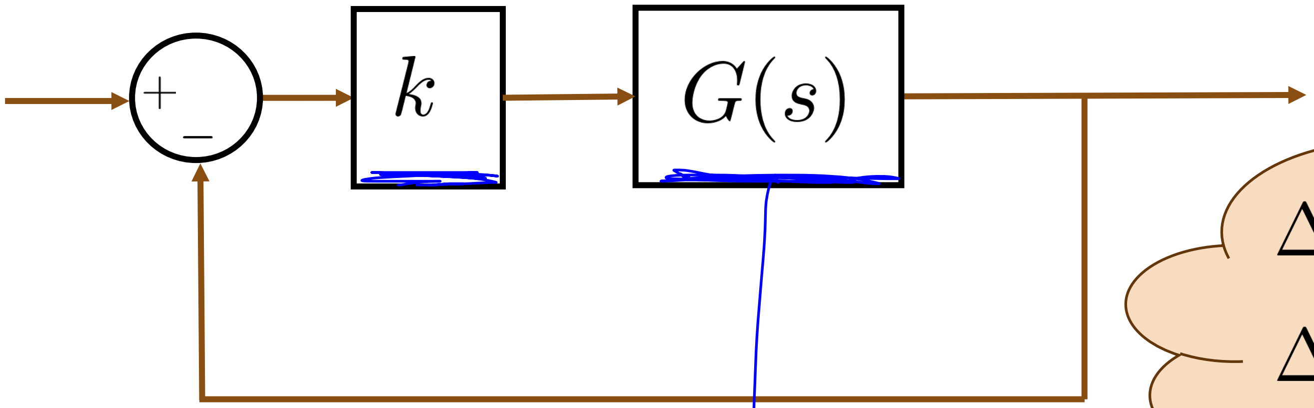


$$G(s)H(s) = \frac{(s+1)^2+2}{(s+1)(s+2)}$$



Stable or Unstable?

Nyquist Criterion



$$\Delta(s) = 1 + kG(s) = 0$$

$$\Delta'(s) = \frac{1}{k} + G(s) = 0$$

stable OL.

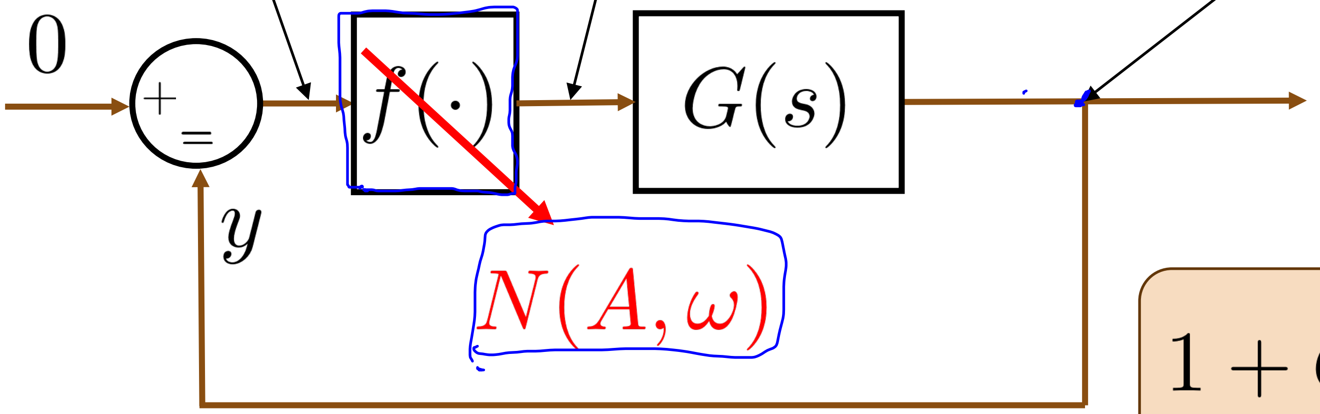
Necessary and *sufficient condition* stability condition for systems for stable open-loop systems:
The Nyquist plot does not encircle the point ***-1/k***

LUND UNIVERSITY

Extension of Nyquist Criterion for Describing Function Analysis (Existence of oscillations)

- Assume that there exists self-sustained oscillations $(1+G(j\omega)N(A,\omega))y=0$

$e=-y$ $\quad$ $w=N(A,\omega)e$ $\quad$ $y=G(j\omega)N(A,\omega)(-y)$



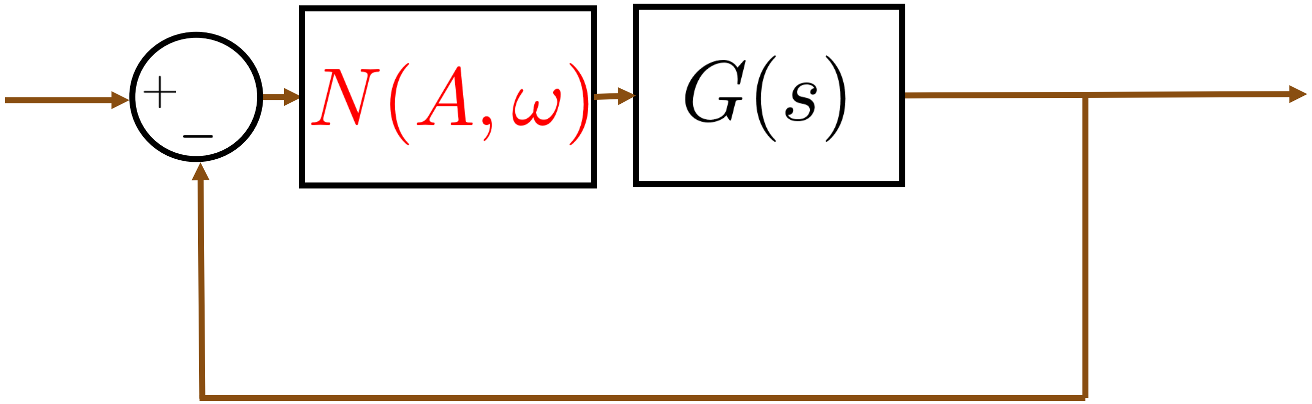
$y(t)\neq 0$

$$\boxed{1+G(j\omega)N(A,\omega)=0\ (*)}$$

- The amplitude and frequency must satisfy (*) – Harmonic balance
- If (*) has no solutions then there are no oscillations in the system

LUND UNIVERSITY

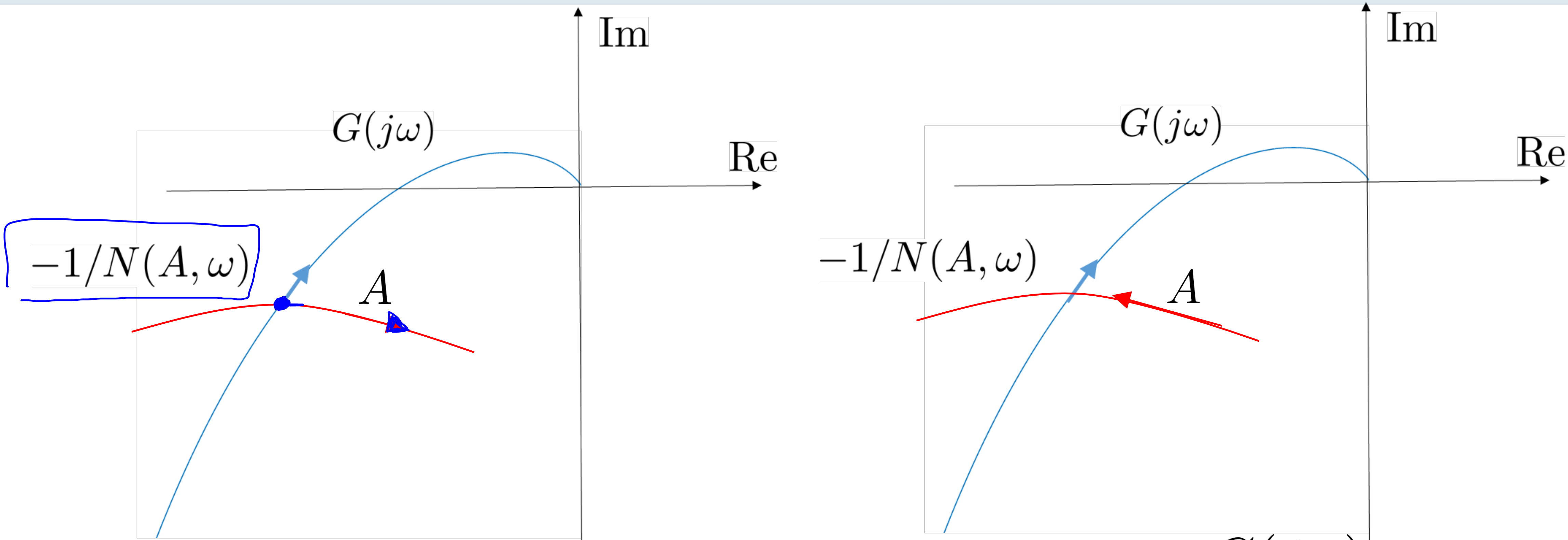
Extension of Nyquist Criterion for Describing Function Analysis (Stability of oscillations)



Necessary and *sufficient condition* stability condition for systems ~~for systems~~ with stable (open-loop) linear part:

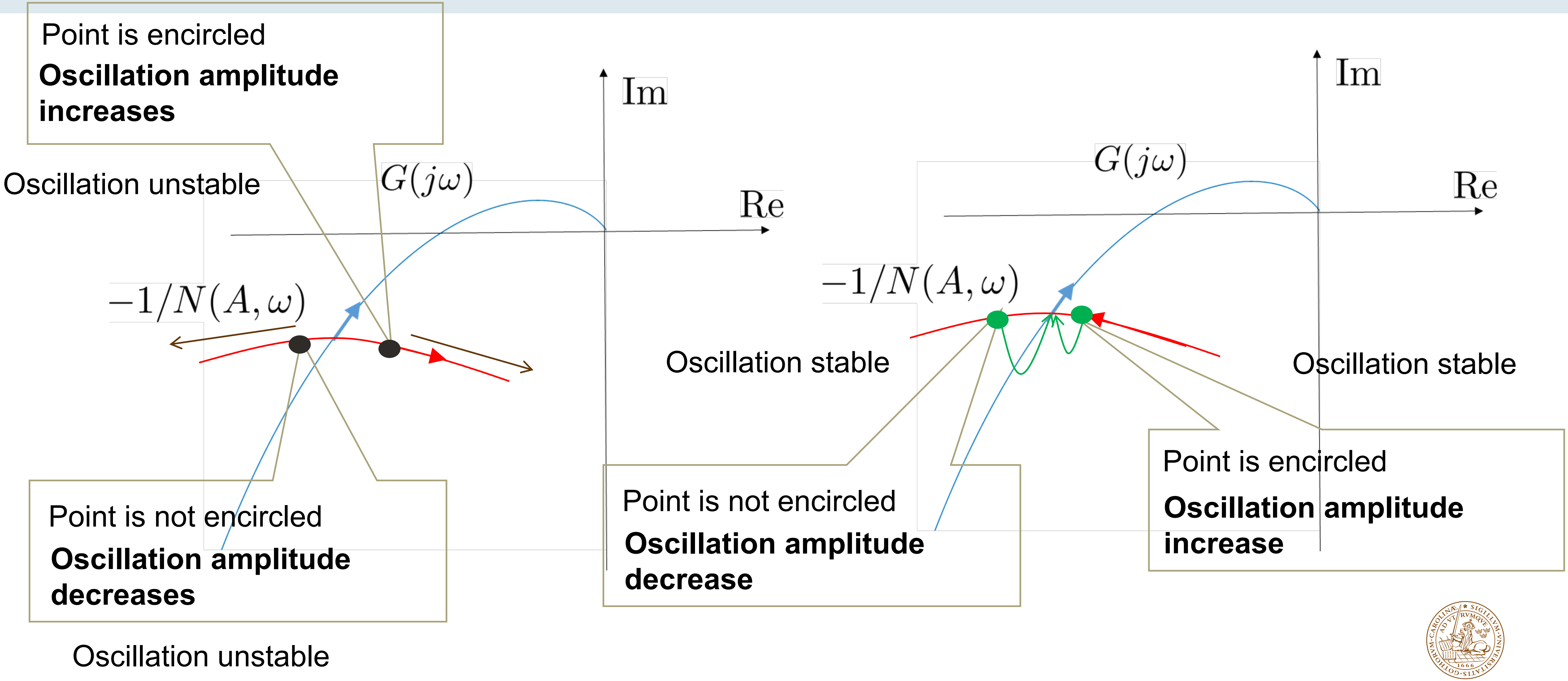
The Nyquist plot does not encircle the point $\dfrac{-1}{N(A,\omega)}$

Stability of oscillations



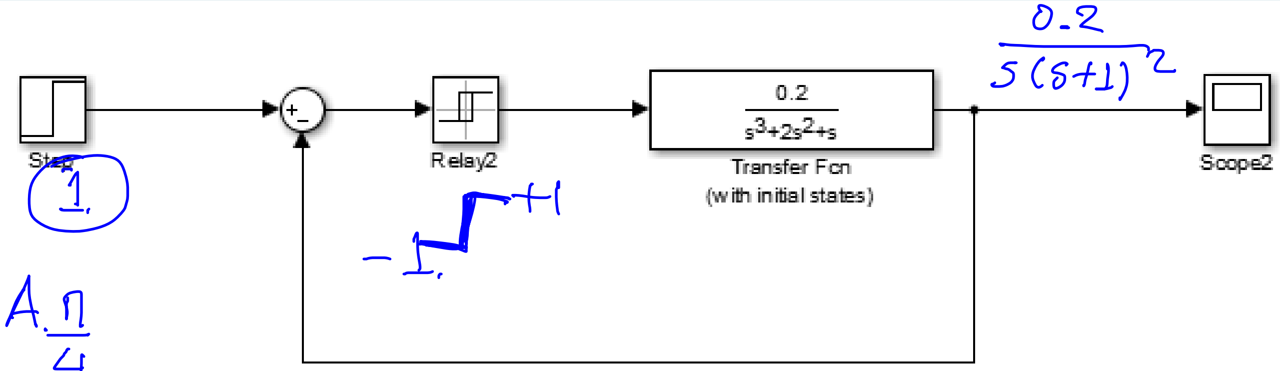
- Permanent oscillation: Common point of the Nyquist diagram $G(j\omega)$ and the plot $\dfrac{-1}{N(A,\omega)}$
- Stability of the oscillation: Does the oscillation continue after a small perturbation in A?

Stability of oscillations



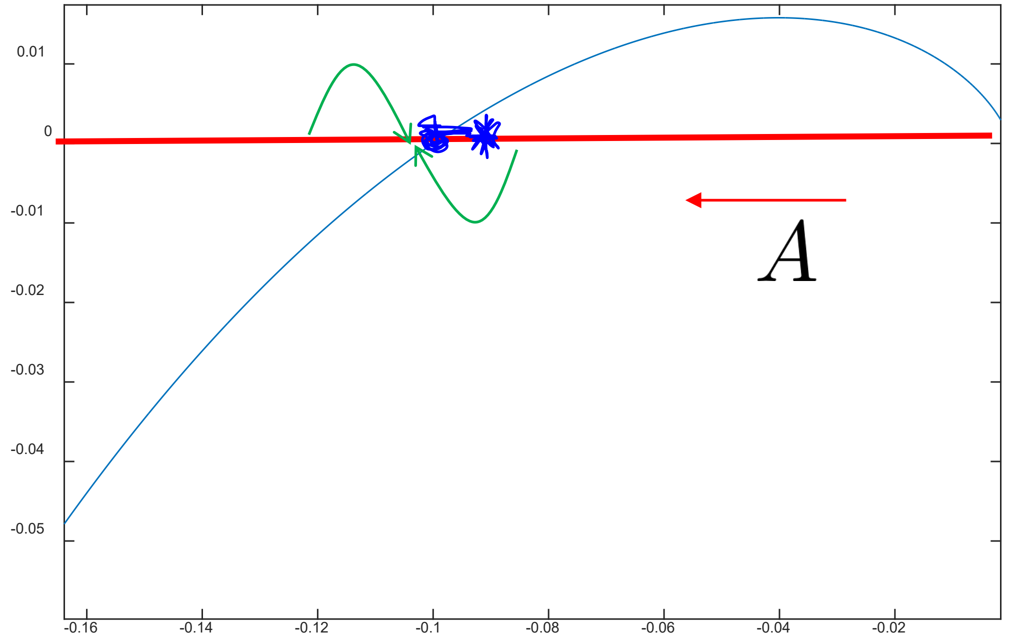
Stability of the oscillation: Does the oscillation continue after a small perturbation in A?

Example – Prediction and stability of persistent oscillations

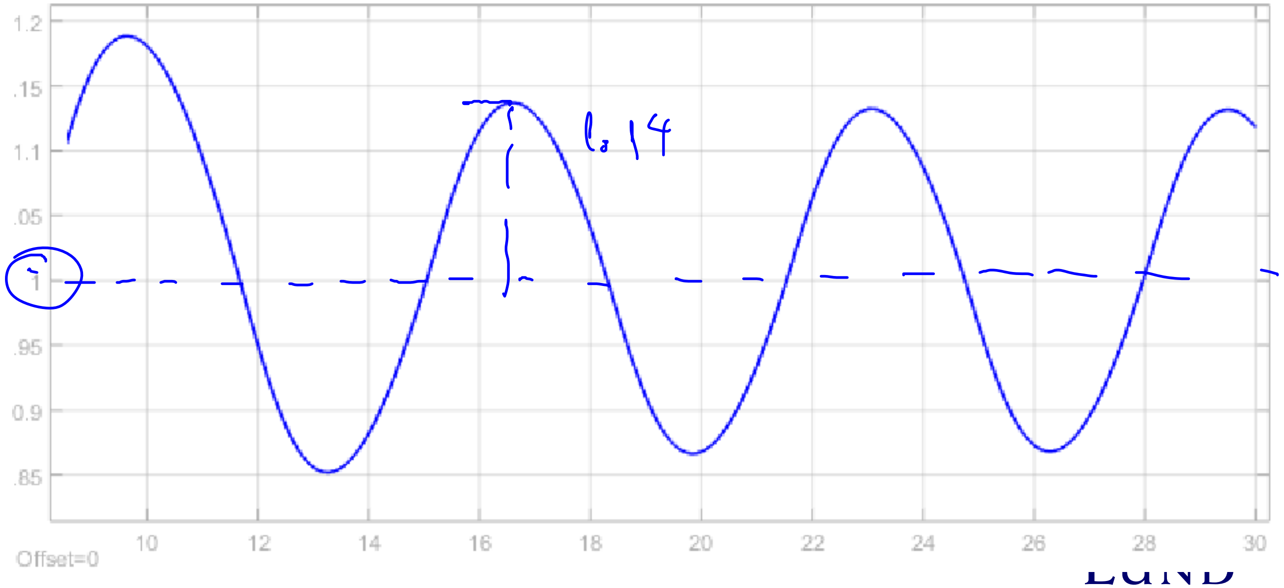


$N = \frac{4}{A\pi} \quad -1/N = -\frac{A\pi}{4}$

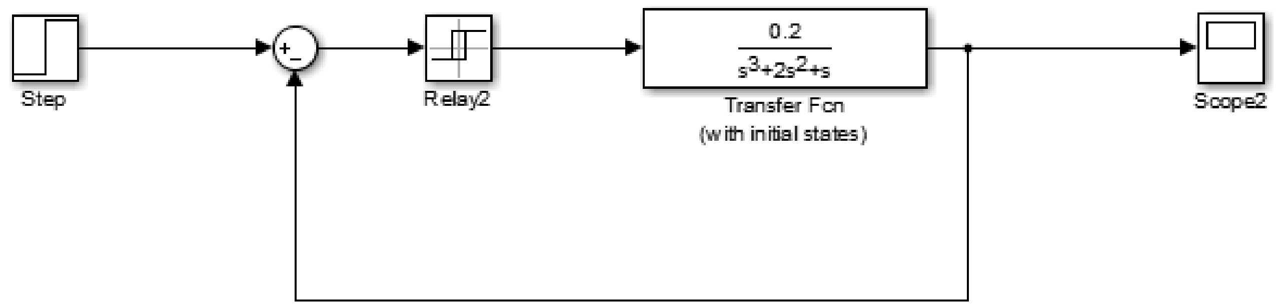
Nyquist Plot – Describing function



System Response



Example – Prediction of oscillations



$$G(s) = \frac{K}{s(1+s)^2}$$

$$K = 0.2$$

$$G(j\omega) = \frac{K}{j\omega(1+j\omega)^2} = \frac{K}{-2\omega^2 + j\omega(1-\omega^2)} \cdot \frac{-2\omega^2 - j\omega(1-\omega^2)}{-2\omega^2 - j\omega(1-\omega^2)}$$

$\underline{s = j\omega}$

$$= \boxed{-\frac{2K\omega^2}{4\omega^4 + \omega^2(1-\omega^2)^2}} - j\boxed{\frac{K\omega(1-\omega^2)}{4\omega^4 + \omega^2(1-\omega^2)^2}}$$

Real for $\boxed{\omega = 1\text{rad/s}}$

$$\left.\begin{aligned} G(j1) &= -\frac{K}{2} \\ N(A) &= \frac{4}{A\pi} \end{aligned}\right\} \Rightarrow A = \frac{2K}{\pi}$$

$$\boxed{G(j\omega) = -\frac{1}{N(A)}}$$

For $K = 0.2$ $\boxed{A = 0.127}$

LUND UNIVERSITY

Describing function analysis: pitfalls

- DF analysis may predict a limit cycle, even if it does not exist.
- A limit cycle may exist, even if DF analysis does not predict it.
- The predicted amplitude and frequency are only approximations and can be far from the true values.