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FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 3: Linearization around a trajectory, Limit cycles and Stability definitions

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Outline

- Linearization around trajectory (general case)
- Limit Cycles
- Definitions of Stability

Material

- Glad& Ljung Ch. 11, 12.1,
  (Khalil Ch 2.3, part of 4.1, and 4.3)
- Lecture slides

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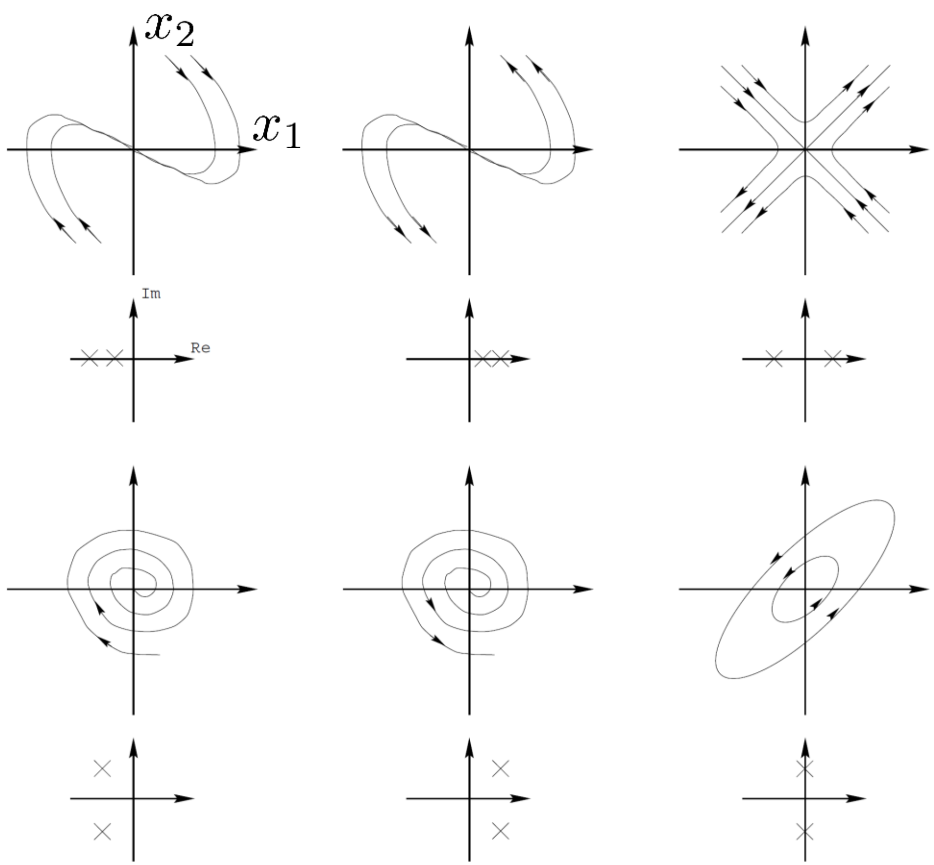
Summary of phase portraits and their equilibriums

	stable node	unstable node	saddle point
$\mathrm{Im}\lambda_i = 0:$	$\lambda_1, \lambda_2 < 0$	$\lambda_1, \lambda_2 > 0$	$\lambda_1 < 0 < \lambda_2$
$\mathrm{Im}\lambda_i \neq 0:$	$\mathrm{Re}\lambda_i < 0$	$\mathrm{Re}\lambda_i > 0$	$\mathrm{Re}\lambda_i = 0$
	stable focus	unstable focus	center point

$\dot{x} = Ax$



Effect of perturbations

Perturbations in $A + \boxed{\Delta}$

- Structurally stable: the qualitative behavior remains the same under arbitrarily small perturbations in A

Examples: a node(with distinct eigenvalues), a saddle or a focus

- A stable node with multiple eigenvalues could become a stable node or a stable focus under arbitrarily small perturbations in A

- A center is not structurally stable

$\Delta = \delta I$

$$\dot{z} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} z \quad \Rightarrow \quad \dot{z} = \begin{bmatrix} \delta & -\omega \\ \omega & \delta \end{bmatrix} z \qquad \boxed{\delta \pm j\omega}$$

Linearization around an equilibrium point

- Linear approximation of nonlinear systems (Taylor expansion)

$$\dot{\tilde{x}} = f(\tilde{x} + x^\star)$$

$$\dot{\tilde{x}} = f(x^\star) + \left.\frac{\partial f(\tilde{x}+x^\star)}{\partial \tilde{x}}\right|_{\tilde{x}=0} \tilde{x} + \text{H.O.T.}$$

$$\tilde{x} = x - x^\star$$

$$f(x) = f(x^\star) + J_f(x^\star)\tilde{x} + \frac{1}{2}\tilde{x}^T H_f(x^\star)\tilde{x} + \cdots$$

Jacobian **Hessian**

$$\dot{\tilde{x}} = f(x^\star) + \frac{\partial f}{\partial x}(x^\star)\tilde{x} + \text{H.O.T.}$$

$$f(x^\star) = 0$$

$$\dot{\tilde{x}} = \frac{\partial f}{\partial x}(x^\star)\tilde{x} + \text{H.O.T.}$$

$$\dot{\tilde{x}} \approx \frac{\partial f}{\partial x}(x^\star)\tilde{x}$$

$$\frac{\partial f}{\partial x}$$

$$J_f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

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Predicting behaviors close to equilibrium

$\tilde{x} = x - x^*$

$A = \left.\frac{\partial f}{\partial x}\right|_{x=x^*}$

$\dot{x} = f(x)$

$\boxed{\dot{\tilde{x}} = A\tilde{x} + g(x),}$

Linear approximation →

$\dot{\tilde{x}} \simeq A\tilde{x}$ *Valid when* $\boxed{\det(A) \neq 0}$

$\lambda_1, \lambda_2 \neq 0$

where $A = \frac{\partial f}{\partial x}(x^\star)$
and $g = f(x) - \frac{\partial f}{\partial x}(x^\star)\tilde{x} \in C^1$ and $\frac{\|g(x)\|}{\|\tilde{x}\|} \to 0$ as $\|\tilde{x}\| \to 0$.

$\dot{x} = f(x)$	$\dot{\tilde{x}} \simeq A\tilde{x}$
???	**Center**
node	node
saddle	saddle
focus	focus
unstable	$\lambda_1,\ \lambda_2 > 0$ → unstable
stable	$\lambda_1,\ \lambda_2 < 0$
???	$\lambda_i = 0$

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Linearization around a trajectory

Idea: Make Taylor-expansion around a known solution $\{x^\star(t), u^\star(t)\}$ satisfying the differential equation:

$$\boxed{\frac{dx^\star}{dt} = f(x^\star(t), u^\star(t))}$$

$\dot{x} = f(x, u)$

Equilibrium points

$\frac{dx^*}{dt} = 0$

$\tilde{x} = x - x^*$

$$\begin{aligned}
\dot{x}(t) &= f(x^\star(t) + \tilde{x}(t), u^\star(t) + \tilde{u}(t)) \\
&= \underbrace{f(x^*(t), u^*(t))}_{\dot{x}^*} + \frac{\partial f}{\partial x}(x^\star(t), u^\star(t))\tilde{x}(t) + \frac{\partial f}{\partial u}(x^\star(t), u^\star(t))\tilde{u}(t) + \boxed{\mathcal{O}(\|\tilde{x}, \tilde{u}\|^2)}
\end{aligned}$$

H.O.T

$$\boxed{\dot{\tilde{x}}(t) = \frac{\partial f}{\partial x}(x^\star(t), u^\star(t))\tilde{x}(t) + \frac{\partial f}{\partial u}(x^\star(t), u^\star(t))\tilde{u}(t) + \mathcal{O}(\|\tilde{x}, \tilde{u}\|^2)}$$

Linearization around a trajectory

Hence, for small $(\tilde{x}, \tilde{u})$, approximately

$$\boxed{\dot{\tilde{x}}(t) = A(t)\tilde{x}(t) + B(t)\tilde{u}(t)}$$

Example: if dim $x = 2$, dim $u = 1$

2x2

$$A(t) = \frac{\partial f}{\partial x}(x^\star(t), u^\star(t)) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} (x^\star(t), u^\star(t))$$

$$B(t) = \frac{\partial f}{\partial u}(x^\star(t), u^\star(t)) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix} (x^\star(t), u^\star(t))$$

2x1

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Linearization around a trajectory

Linearization of the output equation $y(t) = h(x(t), u(t))$ around the nominal output $y^*(t) = h(x^*(t), u^*(t))$:

$\dot{x} = f(x, u)$

$y = h(x, u)$

$$\tilde{y}(t) = \underline{C(t)}\tilde{x}(t) + \underline{D(t)}\tilde{u}(t)$$

Second order system i.e. dim $y =$ dim $x = 2$, dim $u = 1$

$$C(t) = \frac{\partial h}{\partial x}(x^\star(t), u^\star(t)) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix} (x^\star(t), u^\star(t))$$

$$D(t) = \frac{\partial h}{\partial u}(x^\star(t), u^\star(t)) = \begin{bmatrix} \frac{\partial h_1}{\partial u_1} \\ \frac{\partial h_2}{\partial u_1} \end{bmatrix} (x^\star(t), u^\star(t))$$

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Time-varying Linear Systems

Example: Time-varying.

$$\dot{\tilde{x}} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \tilde{x}$$

$\lambda_{1,2} = -1$

By just checking the eigenvalues someone would that the origin is stable.

However, we can solve first for $\tilde{x}_2$ and then substitute in $\dot{\tilde{x}}_1$ and realize the $\tilde{x}_1$ grows unbounded.

$\dot{\tilde{x}}_2 = -\tilde{x}_2 \Rightarrow \tilde{x}_2(t) = \tilde{x}_2(0)\, e^{-t}$

$\dot{\tilde{x}}_1 = -\tilde{x}_1 + \boxed{e^{2t} e^{-t}}\, \tilde{x}_2(0)$

$= -\tilde{x}_1 + e^{t}\, \tilde{x}_2(0)$

$\dot{\tilde{x}}_1$ will grow unbounded

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Time-varying Linear Systems

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t)$$

$\lambda_i[A(t) + A^T(t)] < 0 \ \forall t$ ⇒ Stable

$\lambda_i[A(t) + A^T(t)] > 0 \ \forall t$ ⇒ Unstable

$\lambda_i[A(t) + A^T(t)] = 0 \ \underline{\forall t}$ ⇒ No conclusion

for all $\forall t$

<u>exists</u> $\exists t$

Periodic solutions and Limit Cycles

- A system oscillates when it has a nontrivial periodic solution:

$$x(t+T) = x(t), \quad \forall t \geq 0 \qquad \text{for some} \quad T > 0$$

Example:

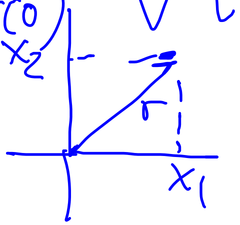
$$\boxed{\dot{z} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} z}$$

polar coordinates

$$\dot{r} = 0 \qquad r(t) = r(0) \quad \forall t \geq 0$$

$$\dot{\theta} = \omega$$

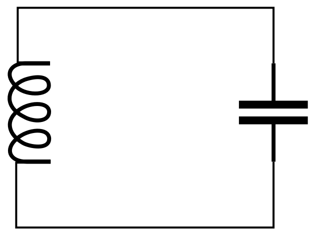
$$\hookrightarrow \theta = \theta_0 + \omega t$$



$$r(0) = \sqrt{x_1^2(0) + x_2^2(0)}$$

The system has a sustained oscillation of amplitude $r(0)$

- Harmonic oscillator LC circuit



  - Small perturbation will destroy the oscillation (e.g. resistance)
  - The amplitude depends on the initial conditions

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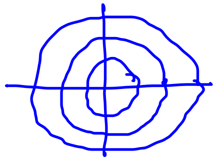
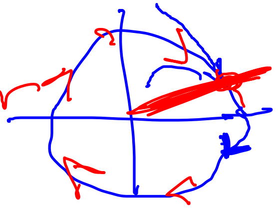
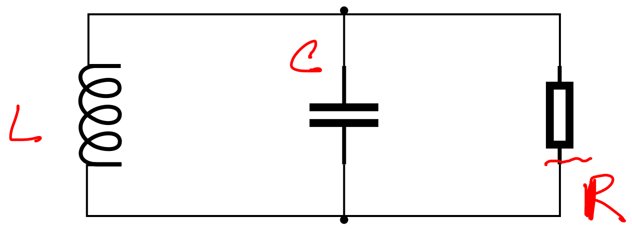
Periodic solutions and Limit Cycles

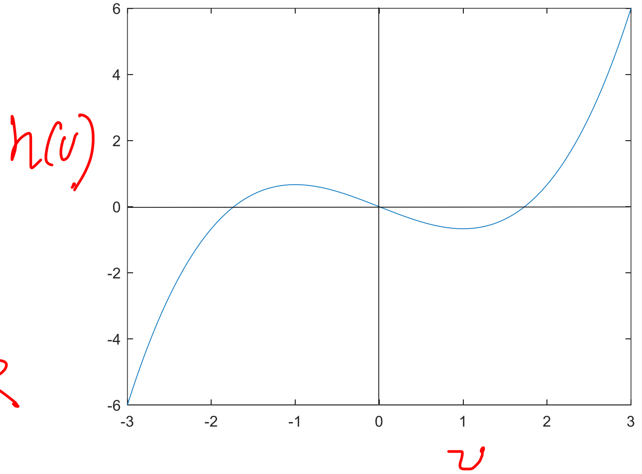
- An **isolated closed** curve in the phase plane
- Closed: periodic solution
- Isolated: limiting nature of the limit cycle, nearby trajectories either converge to or diverge from the limit cycle

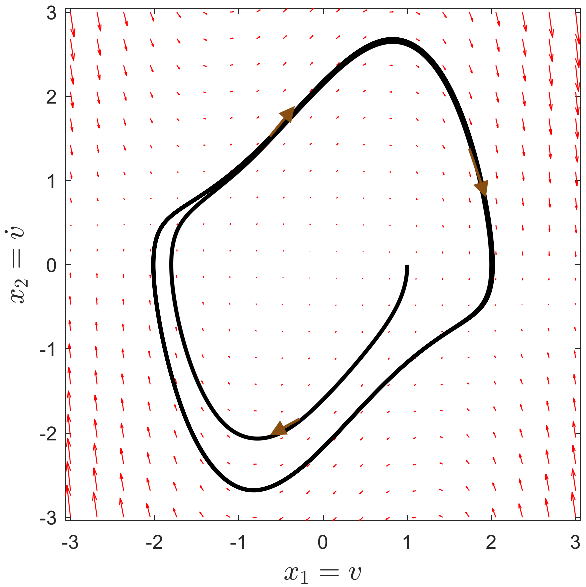




Van der Pol oscillator







$$\ddot{v} - \varepsilon(1 - v^2)\dot{v} + v = 0$$

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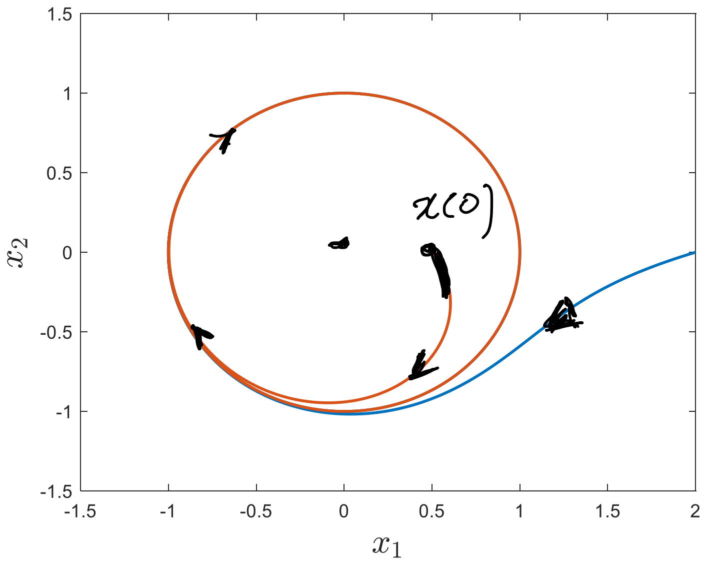
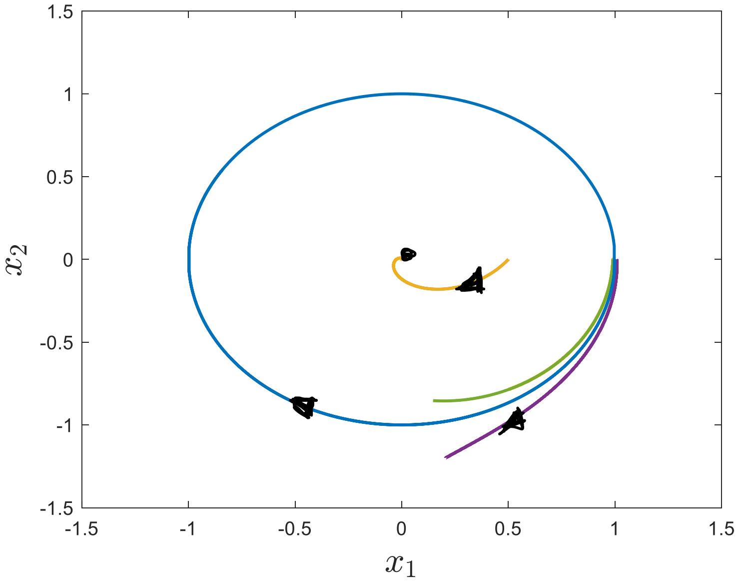
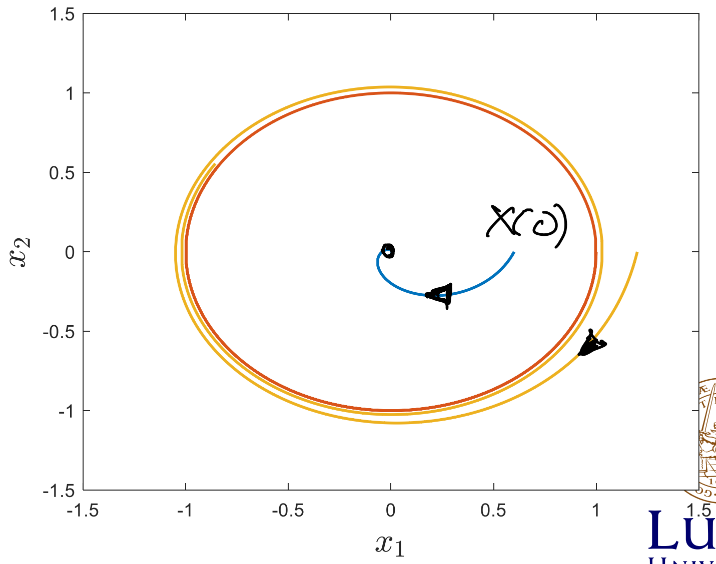
Stability of limit cycles

$\boxed{t \to \infty}$

- **Stable limit cycle:** all trajectories in the vicinity of the limit cycle converge to it
- **Unstable limit cycle:** all trajectories in the vicinity of the limit cycle diverge from it
- **Semi-Stable Limit Cycles:** some of the trajectories in the vicinity converge to it, while the others diverge from it







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Stability of limit cycles – matching quiz

Check first the stability of the eq. (0,0). If it is unstable the system phase portrait corresponds to Fig 1.

$$(A)\ \begin{cases}\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)^2\end{cases} \sim \dot{x} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} x$$

$(\lambda+1)^2 - 1 = 0 \rightsquigarrow \lambda = -1 \pm j$

$$(B)\ \begin{cases}\dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1)\end{cases} \sim \dot{x} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} x$$

$\lambda = -1 \pm j$

$$(C)\ \begin{cases}\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)\end{cases} \sim \dot{x} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} x$$

$(\lambda-1)^2 + 1 = 0 \Rightarrow \lambda = 1 \pm j$

Only the linearization of the "green" (C) system gives unstable eigenvalues ($\mathrm{Re}\,\lambda > 0$)

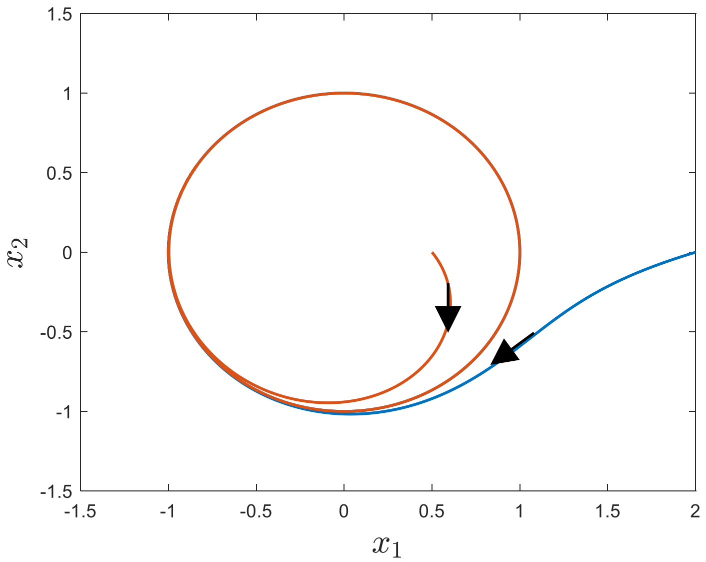

Fig 1. (C)

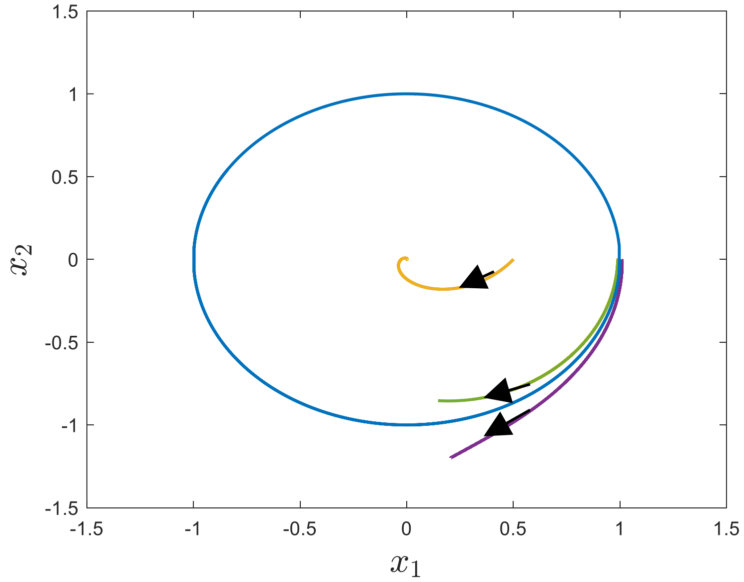

Fig. 2

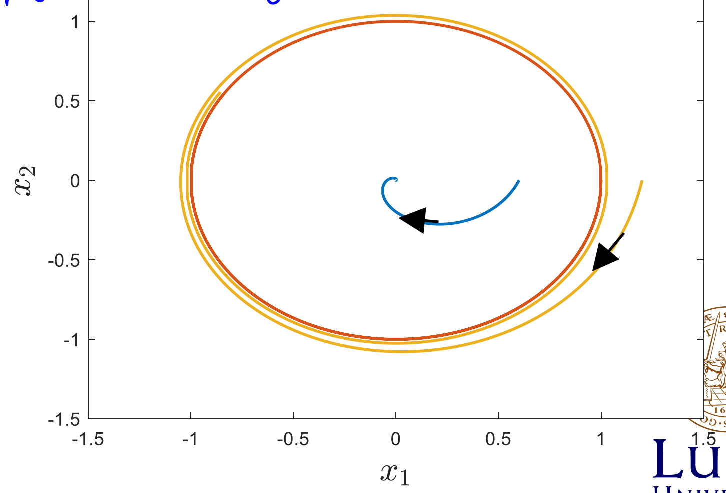

Fig. 3

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Stability of limit cycles (linearization around a trajectory)

- Study the stability of the trajectory $(\sin t, \cos t)$

$$(*)\begin{cases} \dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1) \end{cases}$$

Plug in the trajectory in $(*)$ and verify $\dot{x}^* = f(x^*)$

$x_1 = \sin t$

$x_2 = \cos t$

$\dot{x}_1 = \cos t + 0$

$\frac{d}{dt}(\sin t) = \cos t$

Calculate the Jacobian

$$\frac{\partial f}{\partial x}(x_1, x_2) = \begin{bmatrix} (x_1^2 + x_2^2 - 1) + 2x_1^2 & 1 + 2x_1x_2 \\ 2x_1x_2 - 1 & (x_1^2 + x_2^2 - 1) + 2x_2^2 \end{bmatrix}$$

$$\frac{\partial f}{\partial x}(\sin t, \cos t) = \underbrace{\begin{bmatrix} 2\sin^2 t & 1 + 2\sin t\cos t \\ -1 + 2\sin t\cos t & 2\cos^2 t \end{bmatrix}}_{A}$$

$$A + A^T = \begin{bmatrix} 4\sin^2 t & 4\sin t\cos t \\ 4\sin t\cos t & 4\cos^2 t \end{bmatrix}$$

$\lambda = 0$, $\lambda = 1$.

Since $A(t) = \frac{\partial f}{\partial x}$ is time-dependent we cannot conclude by checking the eigenvalues of $A(t)$

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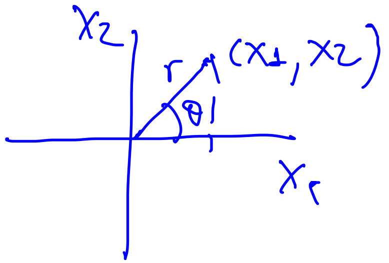
$$\frac{d}{dt}(x_1^2 + x_2^2) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

- Change variables $r = \sqrt{x_1^2 + x_2^2}$ $\theta = \arctan x_2/x_1$



$$\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)^2 \end{cases}$$

$$\boxed{\dot{r} = \frac{1}{\sqrt{x_1^2+x_2^2}}(x_1\dot{x}_1 + x_2\dot{x}_2)}$$

$$\dot{r} = \frac{-1}{\cancel{r}} \cdot r^{\cancel{2}}(r^2-1)^2 = -r(r^2-1)^2$$

$$\dot{\theta} = \frac{1}{x_1^2+x_2^2}(\dot{x}_2 x_1 - x_2\dot{x}_1)$$

Intuition

Otherwise calculate $r(t)$ in closed form

if $r(0) > 1$ $\dot{r}(0) < 0$

r is decreasing but will ~~hit~~ hit $r=1$ and stay there

if $r(0) < 1$ $\dot{r}(0) < 0$

r is decreasing until $r=0$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

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Stability of an equilibrium point

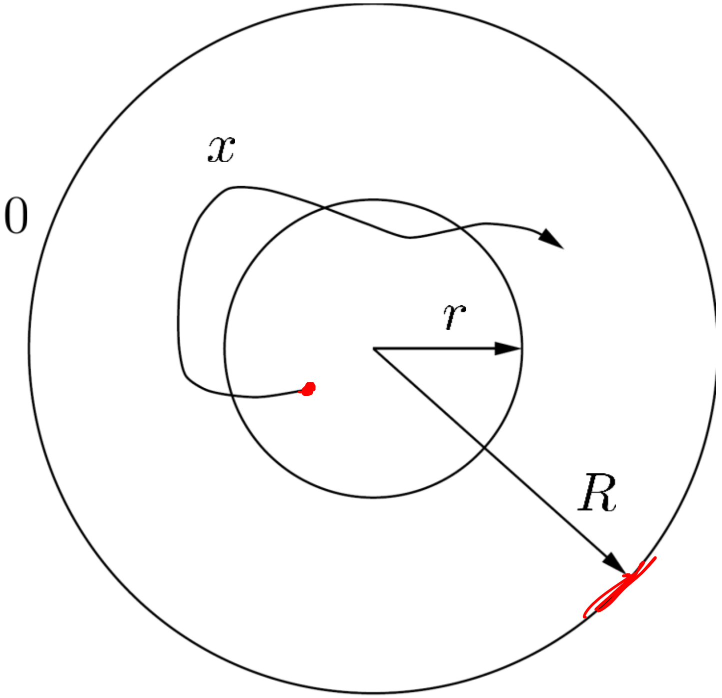
Consider $\dot{x} = f(x)$ where $f(x^*) = 0$

$\forall$ $\exists$

Definition The equilibrium $x^\star$ is **stable** if, for any $R > 0$, there exists $r > 0$ such that

$$\|x(0) - x^\star\| < r \quad \Longrightarrow \quad \|x(t) - x^\star\| < R, \quad \text{for all } t \geq 0$$

Otherwise the equilibrium point $x^\star$ is **unstable**.

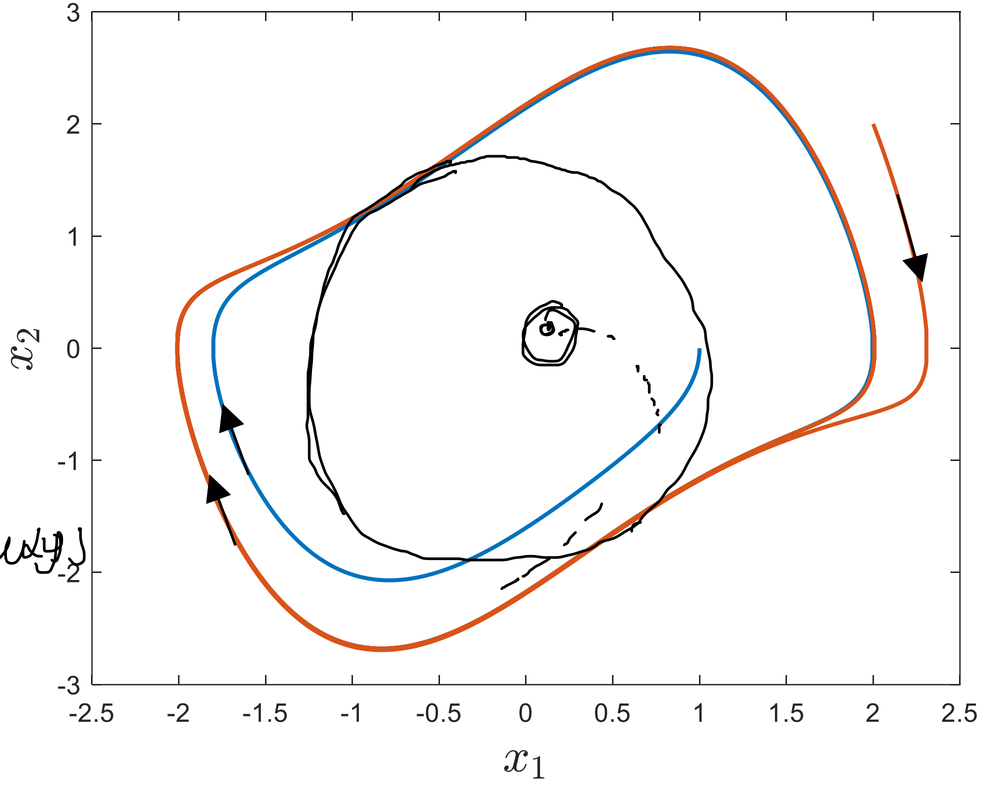


- Use the term “stable (unstable) system” only for linear systems
- A nonlinear system have more than one equilibrium points that each one can be either stable or unstable
- Unstable equilibrium does not mean unbounded trajectories

Unstable equilibrium does not mean unbounded trajectories

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = (1 - x_1^2)x_2 - x_1$$

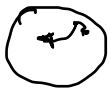
For R defining cycles in the limit cycle the trajectory will always escape R even if it starts very close to equilibrium.



Local vs Global Stability of Equilibrium

Definition The equilibrium $x^\star$ is **locally asymptotically stable (LAS)** if it

1) is stable

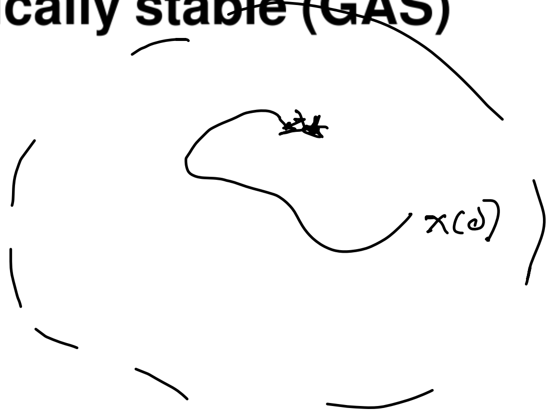


2) there exists $r > 0$ so that if $\|x(0) - x^\star\| < r$ then

$$x(t) \longrightarrow x^\star \quad \text{as} \quad t \longrightarrow \infty.$$

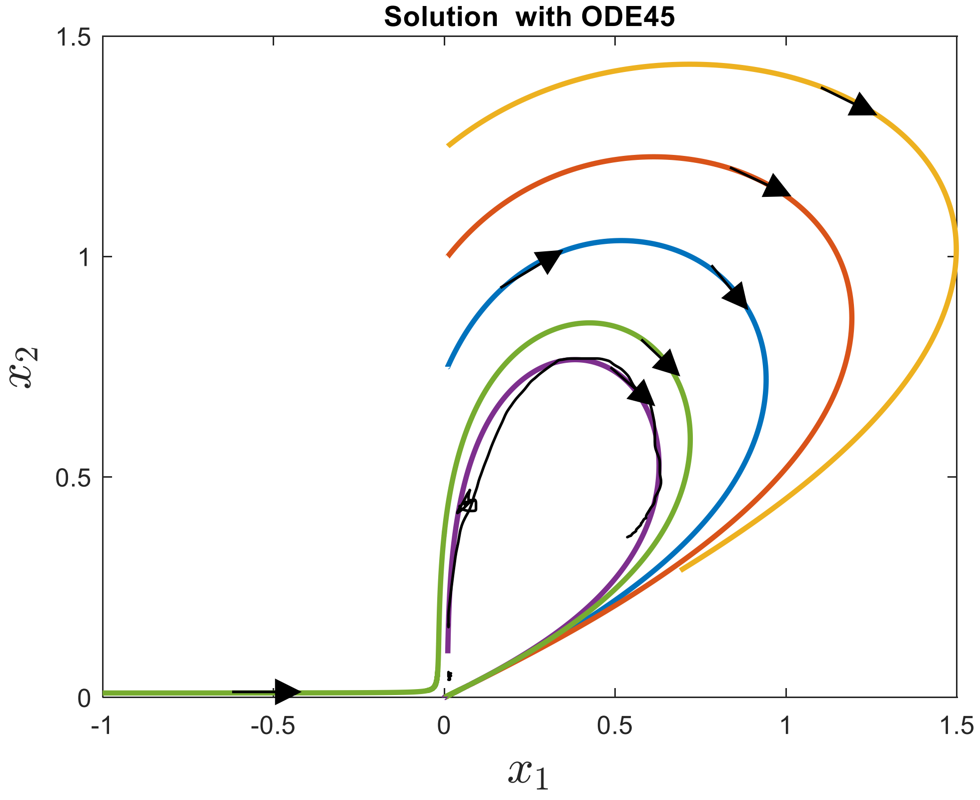
Definition The equilibrium is said to be **globally asymptotically stable (GAS)** if it is LAS and for all $x(0)$ one has

$$x(t) \rightarrow x^\star \text{ as } t \rightarrow \infty.$$



Convergent but not stable

$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)[1 + (x_1^2 + x_2^2)^2]}$$
$$\dot{x}_2 = \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)[1 + (x_1^2 + x_2^2)^2]}$$



Exponential stability

Definition The equilibrium is said to be **exponentially stable (ES)** if there exists to positive constants a and λ such that:

$$\|x(t) - x^\star\| \leq a\|x(0) - x^\star\|e^{-\lambda t}, \quad \text{for all } t \geq 0$$

Global: For all $x(0)$

Local: For $\|x(0) - x^\star\| < r$