FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 2: Linearization and Phase plane analysis

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Outline

- Linearization around equilibrium
- Phase plane analysis of linear systems

Material

- Glad and Ljung: Chapter 13
- Khalil: Chapter 2.1–2.3
- Lecture notes

Linearization around an equilibrium point

- Linear systems with non-zero equilibrium points

Change of variables to move the origin to the equilibrium point

Example $\boxed{\dot{x} = Ax} + b$ $\qquad \dot{x} \equiv 0$

$Ax + b = 0$ Equilibrium $x^\star = -A^{-1}b$

A full rank

New variable $\boxed{\tilde{x}} = x - x^\star$

$$x = \tilde{x} + x^\star$$

$$\dot{\tilde{x}} = \dot{x} + \cancel{\dot{x}^\star} \rightarrow 0 \quad \text{constant}$$

$$\boxed{\dot{\tilde{x}} = A\tilde{x}}$$

$$f^{(n)} = \frac{\partial^n f}{\partial x^n}$$

- Linear approximation of nonlinear systems (Taylor expansion $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$)

$$\dot{x} = f(x)$$

$\dot{x} \equiv 0 \rightarrow$ Equilibrium $f(x^\star) = 0$

New variable $\tilde{x} = x - x^\star$

$$\dot{\tilde{x}} = \dot{x}$$

$$x = \tilde{x} + x^\star$$

$$\boxed{\dot{\tilde{x}} = f(\tilde{x} + x^\star)}$$

Linearization around an equilibrium point

- Linear approximation of nonlinear systems (Taylor expansion)

$$\dot{\tilde{x}} = f(\tilde{x} + x^\star) \quad \longrightarrow \quad \dot{\tilde{x}} = f(x^\star) + \left.\frac{\partial f(\tilde{x}+x^\star)}{\partial \tilde{x}}\right|_{\tilde{x}=0} \tilde{x} + \text{H.O.T.}$$

$$\tilde{x} = x - x^\star$$

$$\boxed{f(x)} = f(\cancel{x}^{\star}) + J_f(x^\star)\tilde{x} + \frac{1}{2}\tilde{x}^T H_f(x^\star)\tilde{x} + \cdots$$

Jacobian **Hessian**

$$\begin{bmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{bmatrix}$$

e.g. $f(x) = \begin{bmatrix} x_2 \\ f_2(x_1, x_2) \end{bmatrix}$

$$\dot{\tilde{x}} = f(\cancel{x^\star}) + \frac{\partial f}{\partial x}(x^\star)\tilde{x} + \text{H.O.T.}$$

Equilibrium

$$\boxed{f(x^\star) = 0}$$

$$\dot{\tilde{x}} = \underbrace{\frac{\partial f}{\partial x}(x^\star)}_{J_f}\tilde{x} + \text{H.O.T.}$$

$$\dot{\tilde{x}} \approx \frac{\partial f}{\partial x}(x^\star)\tilde{x}$$

$$J_f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$J_f = \begin{bmatrix} 0 & 1 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

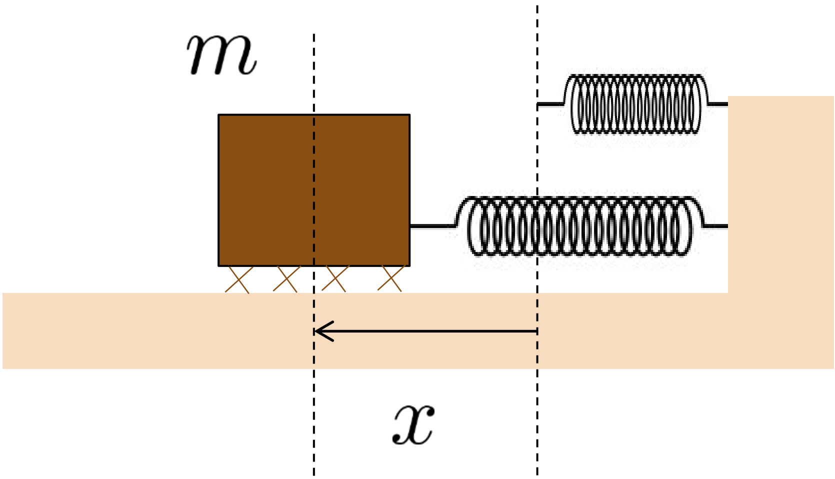
for small scalar $\tilde{x}$

$f(\tilde{x}) = 2\tilde{x} + \tilde{x}^2$

$\tilde{x}^2 \ll \tilde{x}$

$\Rightarrow f(\tilde{x}) \simeq 2\tilde{x}$

Example (nonlinear spring with external force)



- Differential Equation

$$m\ddot{x} + k_v\dot{x} + k_s x^3 = F$$

- State space representation

Position: $x_1 = x$ **Velocity:** $x_2 = \dot{x}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k_s}{m}x_1^3 - \frac{k_v}{m}x_2 + \frac{F}{m}$$

- State space representation (vector form) $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$

$$\dot{x} = f(x) \quad \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} f(x) = \begin{bmatrix} x_2 \\ -\frac{k_s}{m}x_1^3 - \frac{k_v}{m}x_2 + \frac{F}{m} \end{bmatrix}$$

$\dot{x}_1 = f_1(x) = x_2$

$\dot{x}_2 = f_2(x) = -\frac{k_s}{m} x_1^3 - \frac{k_v}{m} x_2 + \frac{F}{m}$

Equilibrium

$\dot{x}_1 = 0$

$\dot{x}_2 = 0$

$$\begin{cases} x_2 = 0 \\ k_s x_1^3 = F \Rightarrow x_1^3 = \frac{F}{k_s} \Rightarrow x_1 = \left(\frac{F}{k_s}\right)^{1/3} \end{cases}$$

Jacobian

$$J_f = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{3k_s x_1^2}{m} & -\frac{k_v}{m} \end{bmatrix}$$

$$J_f(x^*) = J_f\Big(\underbrace{\left(\frac{F}{k_s}\right)^{1/3}}_{x_{1,eq}}, \underbrace{0}_{x_{2,eq}}\Big)$$

Jacobian calculated at equil.

$$J_f(x^*) = \frac{\partial f}{\partial x}(x^*) =$$

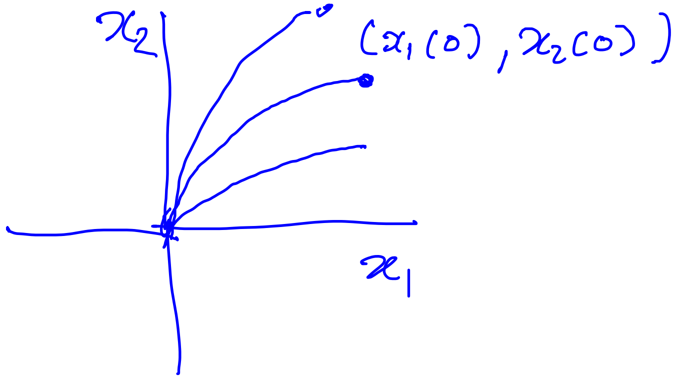
$$\begin{bmatrix} 0 & 1 \\ -\frac{3k_s}{m}\left(\frac{F}{k_s}\right)^{2/3} & -\frac{k_v}{m} \end{bmatrix}$$

Phase plane analysis

- The phase plane method is the graphical study of second-order autonomous systems:

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

$\Rightarrow x_1(t)$
$x_2(t)$

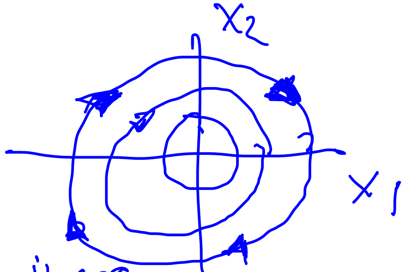


- Phase plane has x_1 and x_2 as coordinates.
- Phase plane trajectory: a curve of the phase plane representing the solution for initial conditions $x_1(0),\ x_2(0)$ with time t varied from 0 to infinity
- Phase portrait: a family of phase plane trajectories from various initial conditions
- Example: $\ddot{y} + y = 0$

$x_1 = y(t) = c\sin(t + \phi_0)$
$x_2 = \dot{y}(t) = c\cos(t + \phi_0)$

$x_1^2 + x_2^2 = c$

$\dot{x}_1 = x_2$
$\dot{x}_2 = -x_1$



for $x_2 > 0$
$\dot{x}_1 > 0 \Rightarrow x_1$ increases

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A first glimpse on phase portraits

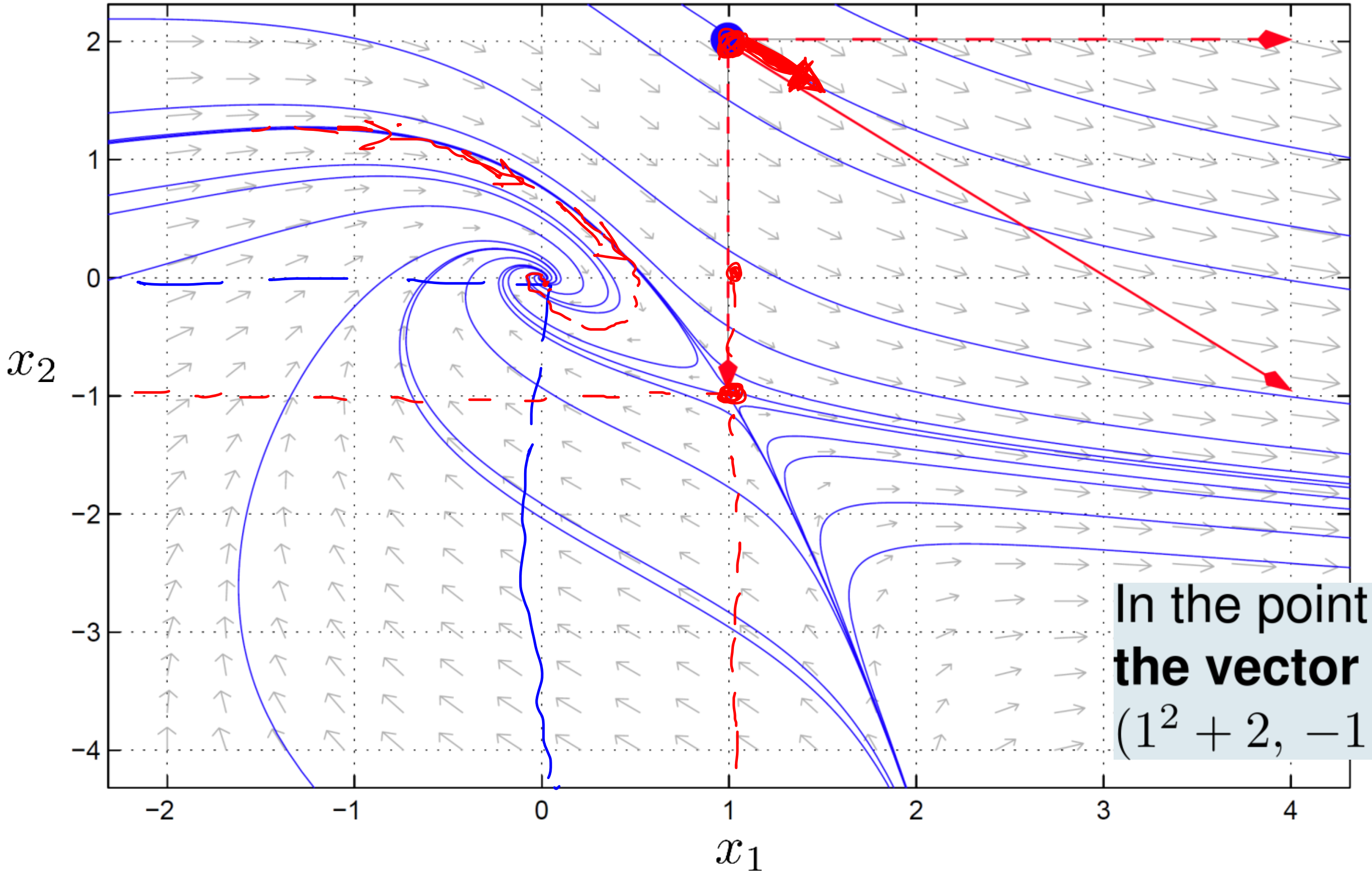
$$\dot{x}_1 = f_1(x_1, x_2) = x_1^2 + x_2$$
$$\dot{x}_2 = f_2(x_1, x_2) = -x_1 - x_2$$

Eq
$x_1^2 + x_2 = 0 \overset{(*)}{\Rightarrow} x_1^2 - x_1 = 0$ → $x_1 = 0$, $x_2 = 0$; $x_1 = 1$, $x_2 = -1$
$-(x_1 + x_2) = 0 \Rightarrow x_2 = -x_1$ (*)

- The **vector field** $f(x_1, x_2) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$ is tangent at point (x_1, x_2) because

$$\frac{dx_2}{dx_1} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$$

- The slope is indeterminate at equilibrium points aka singular points
- Don't forget the arrows!



In the point $(x_1,\ x_2) = (1,\ 2)$
the vector field is pointing in the direction
$(1^2 + 2,\ -1 - 2) = (3,\ -3)$.

Solution of Linear Systems of diff. eq.

$\dot{x} = \alpha x \Rightarrow x(t) = x(0)\, e^{\alpha t}$

State space representation: $\dot{x} = \boxed{A}x$

similar to scalar

Solution: $\boxed{\dot{x} = e^{At} x(0)}$

$z = Wx$

Similarity transformation and change of variables: $\dot{x} = Ax \Rightarrow \dot{x} = W^{-1}MWx \Rightarrow \boxed{\dot{z} = Mz}$

Real distinct eigenvalues $\lambda_1,\ \lambda_2$

$$M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

One double eigenvalue λ

$$M = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Complex Eigenvalues $\sigma \pm j\omega$

$$M = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix}$$

Two real eigenvectors

$$W = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

Eigenvalues: $\det(\lambda I - A) = 0$

Eigenvectors: $Av = \lambda v$

$$W = \begin{bmatrix} v & R_{\frac{\pi}{2}} v \end{bmatrix}$$

Rotation matrix $R = \begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$

Complex eigenvectors

$$W = \begin{bmatrix} \Re(v_1) & \Im(v_1) \end{bmatrix}$$

Real

Imaginary

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Solution of Linear Systems of diff. eq.

State space representation after change of variables $z = W^{-1}x$: $\dot{z} = Mz$

Solution for the new state: $z(t) = e^{Mt}z(0)$

Real distinct eigenvalues $e^{Mt} = \mathrm{diag}(e^{\lambda_1 t}, e^{\lambda_2 t})$ $W = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

One double eigenvalue $e^{Mt} = \mathrm{diag}(e^{\lambda t} + te^{\lambda t}, e^{\lambda t})$ $W = \begin{bmatrix} v & R_{\frac{\pi}{2}}v \end{bmatrix}$

Complex Eigenvalues $e^{Mt} = e^{\sigma t I}e^{\omega t S} = \boxed{e^{\sigma t}}\underline{R_{\omega t}}$ $W = \begin{bmatrix} \Re(v_1) & -\Im(v_1) \end{bmatrix}$

$\hookrightarrow \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}$

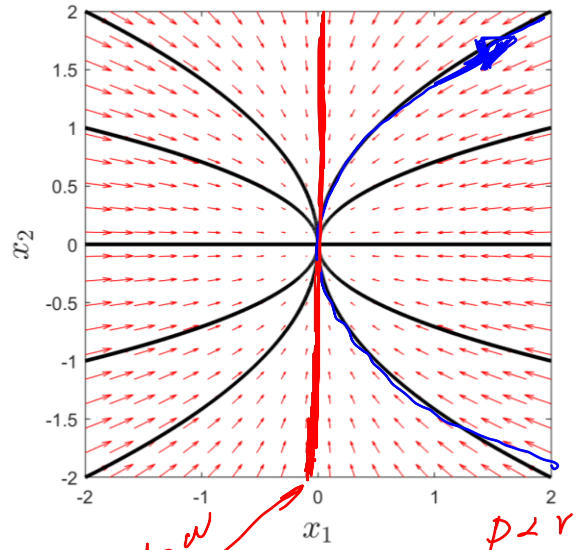
Solution of the original state: $x(t) = W\cancel{^{-1}}z(t) = W\cancel{^{-1}}e^{Mt}z(0)$

Two real eigenvalues

Direct elimination of time variable | Separation of variables | Integration

$$\begin{aligned}\dot{x}_1 &= \lambda_1 x_1\\ \dot{x}_2 &= \lambda_2 x_2\end{aligned} \;\Longrightarrow\; \frac{dx_1}{dx_2} = \underbrace{\left(\frac{\lambda_1}{\lambda_2}\right)}_{\gamma}\frac{x_1}{x_2} \;\Longrightarrow\; \frac{dx_1}{x_1} = \gamma\frac{dx_2}{x_2} \;\Longrightarrow\; \boxed{x_1 = Cx_2^{\gamma}}$$

if $\lambda_1 > 0$



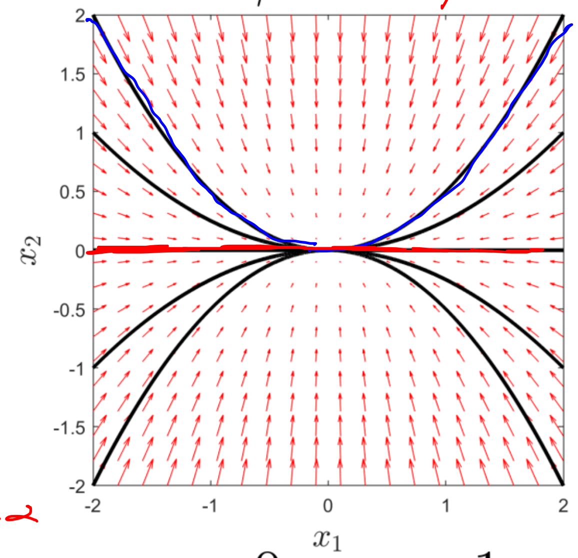
slow

parabola

$\gamma > 1$

$\underbrace{\lambda_1}_{\text{faster}} < \underbrace{\lambda_2}_{\text{slower}} < 0 \qquad 0 < \underbrace{\lambda_2}_{\text{slower}} < \underbrace{\lambda_1}_{\text{faster}}$

parabola

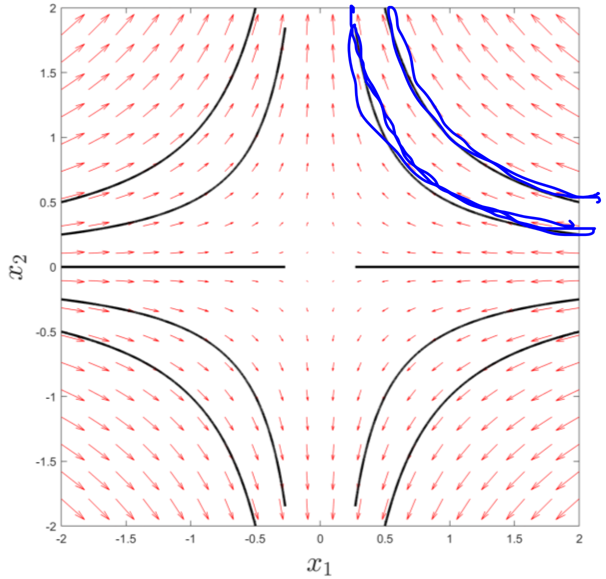


$0 < \gamma < 1$

$x_1 = Cx_2^{\gamma}$

$\Rightarrow \; x_2 = \left(\frac{1}{C}x_1\right)^{1/\gamma}$

Hyperbola



$\gamma < 0$

$\lambda_2 < 0 < \lambda_1$

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Two real eigenvalues

$$\underbrace{\lambda_1}_{\text{faster}} < \underbrace{\lambda_2}_{\text{slower}} < 0$$

v_1 $\quad$ v_2

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

Fast eigenvector

$$x(t) \approx c_1 e^{\lambda_1 t} v_1 + c_2 v_2 \ \boxed{\text{for small } t}$$

Slow eigenvector

$$x(t) \approx c_2 e^{\lambda_2 t} v_2 \text{ for large } t$$

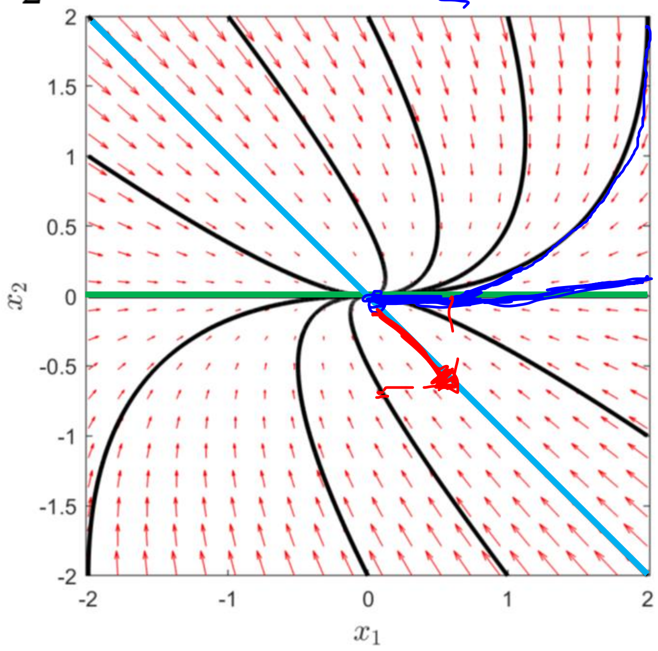
Example:

[X Y] = eigs (A)

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}}^{A} x$$

$$(\lambda_1, \lambda_2) = (-1, -2)$$

$$v_2 = \frac{\sqrt{2}}{2}[1 \ -1]^T$$ -2 faster



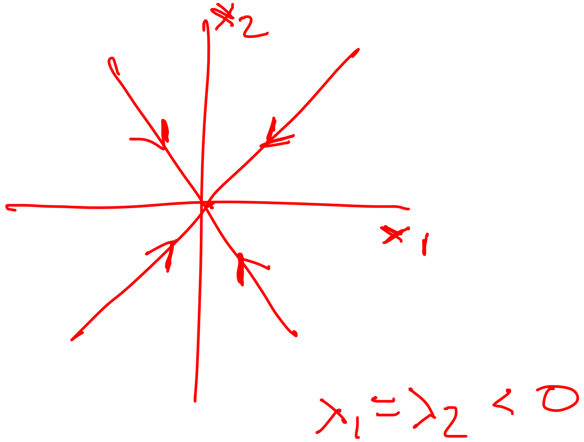
-1

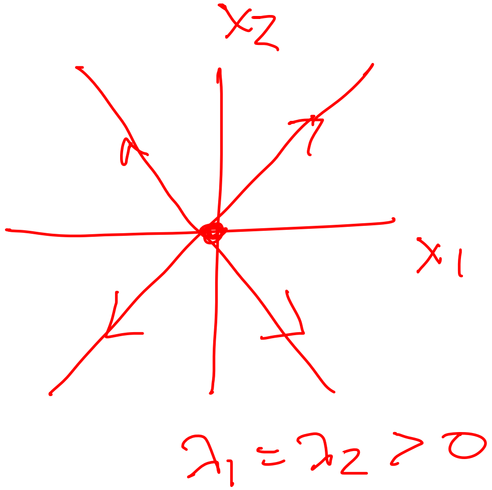
$$v_1 = [1 \ 0]^T$$ slower

Some comments

- What if $\lambda_1 = \lambda_2$?

$x_1 = c\, x_2^{\cancel{\gamma}\, 1}$





Star

- $$\lambda_1 \lambda_2 = \det(A)$$

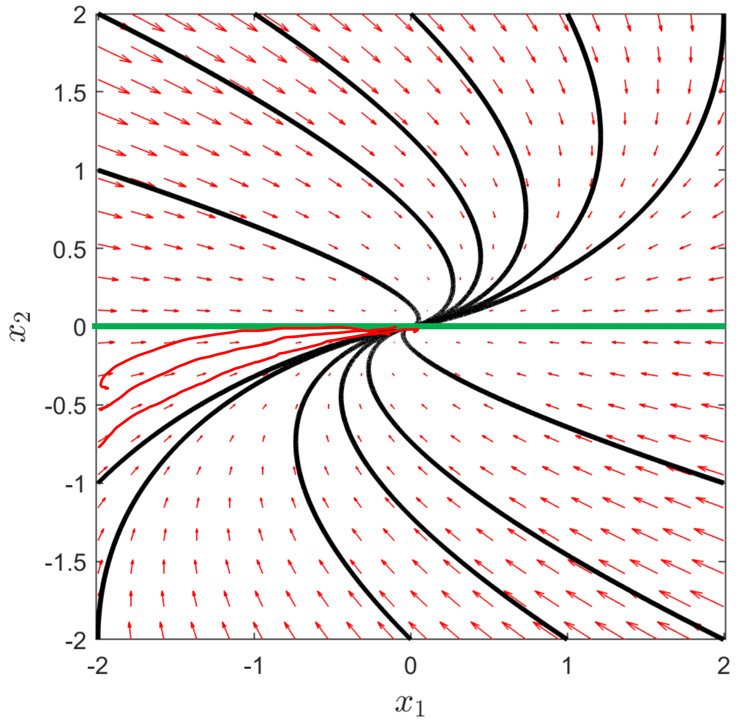
$$\lambda_1 + \lambda_2 = \mathrm{Tr}(A)$$

$$\lambda_{1,2} = \frac{1}{2}\left[\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}^2(A) - 4\det(A)}\right]$$

One tangent mode

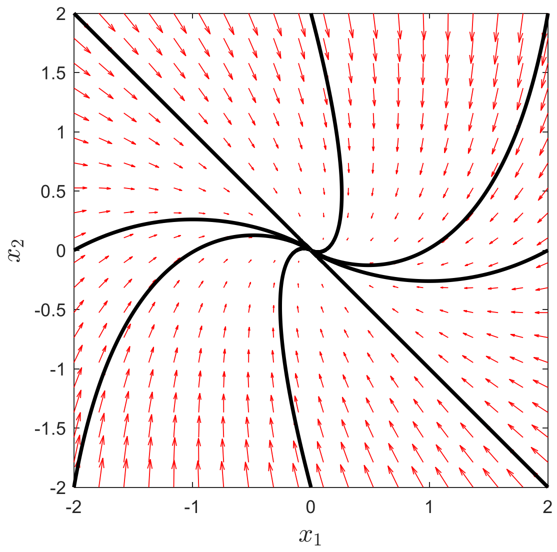
$$\dot{x} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} x, \qquad \operatorname{rank}(\lambda I - A) = 1$$

$$x_1(t) = x_1(0)e^{\lambda t} + t x_2(0)e^{\lambda t}$$
$$x_2(t) = x_2(0)e^{\lambda t}$$

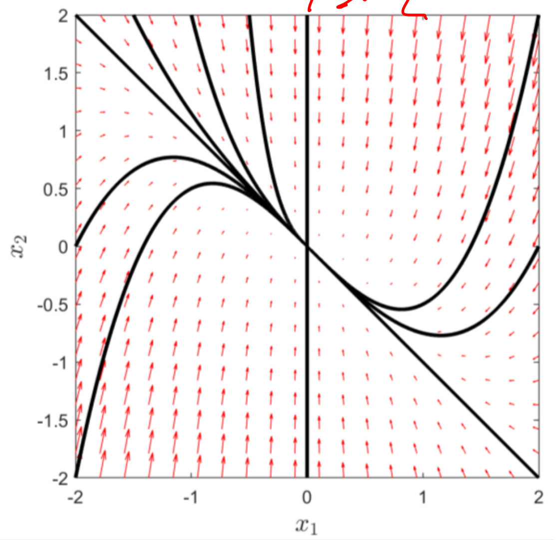


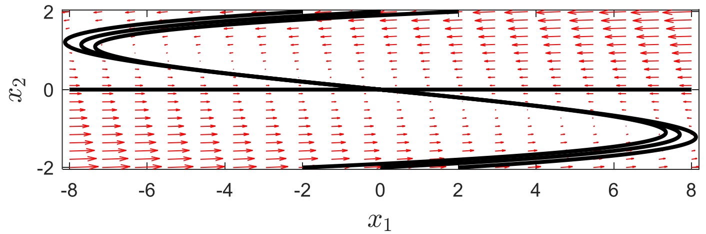
$$v_1 = [1\ 0]^T$$

Matching quiz 1



λ_1, λ_2





$$\dot{x} = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & -1.5 \end{bmatrix} x$$

$$\dot{x} = \begin{bmatrix} -1 & -10 \\ 0 & -1 \end{bmatrix} x$$

$\dot{x}_2 = -x_2.$

$$\dot{x} \begin{bmatrix} -1 & 0 \\ -2 & -3 \end{bmatrix} x$$

Complex eigenvalues

$$z = W^{-1}x$$

$$\dot{x} = Ax \quad \Longrightarrow \quad \dot{z} = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} z$$

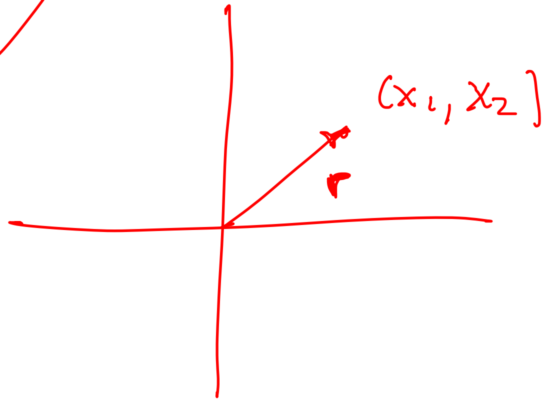
$$\sigma \pm j\omega$$

$$W = \begin{bmatrix} \Re(v_1) & \Im(v_1) \end{bmatrix}$$

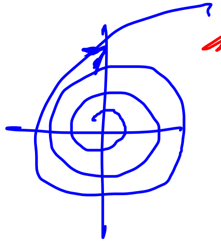
$$r = \sqrt{z_1^2 + z_2^2}, \; \theta = \arctan z_2/z_1 \quad \Longrightarrow \quad \boxed{\begin{aligned} \dot{r} &= \sigma r \\ \dot{\theta} &= \omega \end{aligned}}$$

$$z_1 = r\cos\theta, \; z_2 = r\sin\theta$$

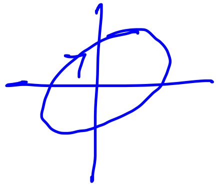
Why?



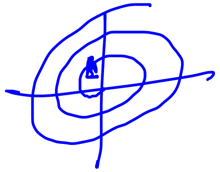
If $\sigma > 0$ Unstable focus $\dot{r} = 6r$



If $\sigma = 0$ Center $\dot{r} = 0$

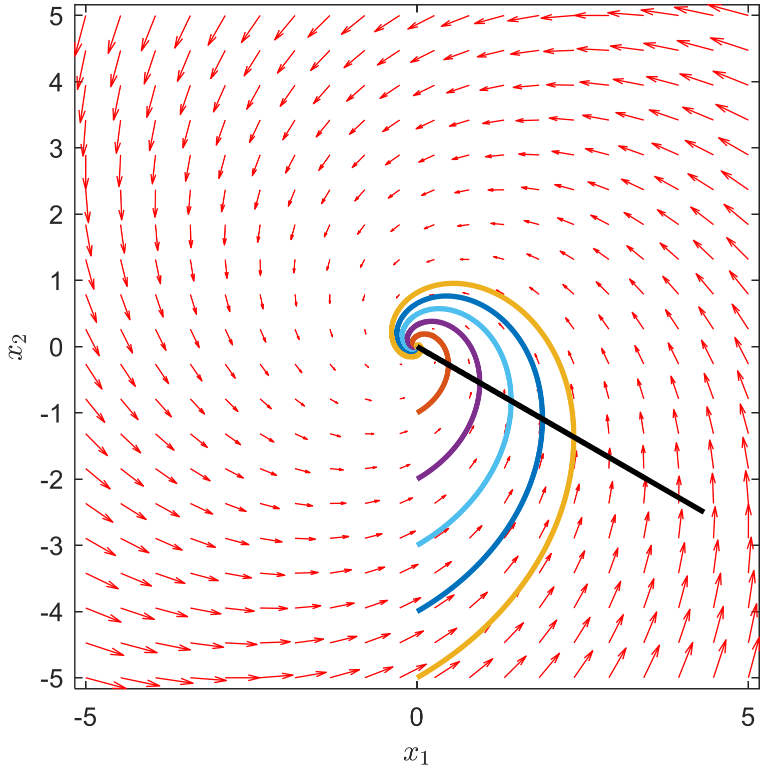


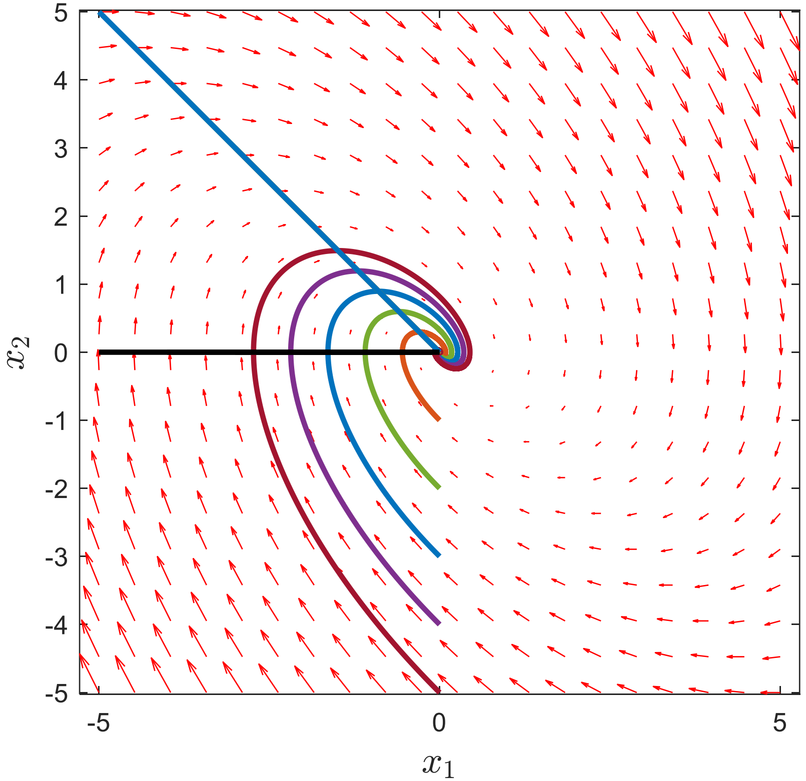
If $\sigma < 0$ Stable focus $\dot{r} = -2r$ $r \to 0$



Matching quiz 2

$$\dot{x} = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} x \qquad \lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j \qquad \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x$$

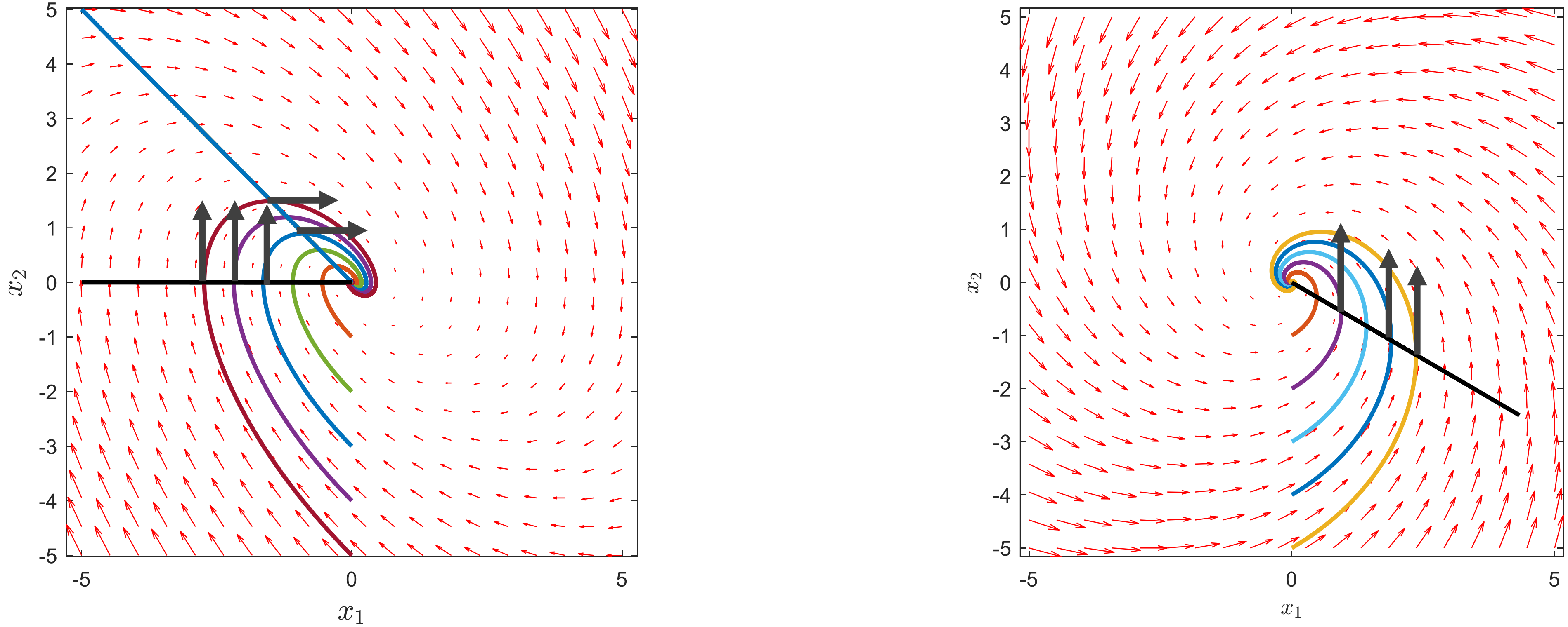




Example

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x \qquad \lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j \quad \longrightarrow \quad \dot{z} = \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} z \quad \longrightarrow \quad \begin{aligned} \dot{r} &= -\frac{1}{2} r \\ \dot{\theta} &= \frac{\sqrt{3}}{2} \end{aligned}$$

- Stable focus $\quad \sigma = -1/2 < 0$



How to draw phase portraits

If done by hand then

1. Find equilibria (also called singularities)
2. Sketch local behavior around equilibria
3. Sketch $(\dot{x}_1, \dot{x}_2)$ for some other points. Use that $\frac{dx_1}{dx_2} = \frac{\dot{x}_1}{\dot{x}_2}$.
4. Try to find possible limit cycles
5. Guess solutions

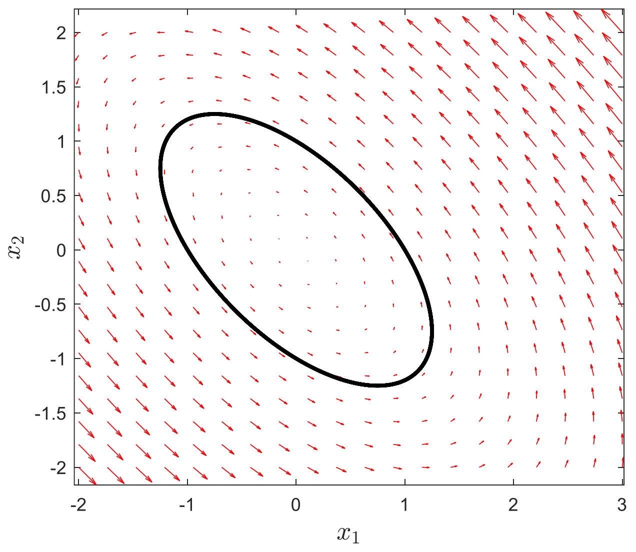
Matlab: PPTool and some other tools for Matlab is available on Canvas.

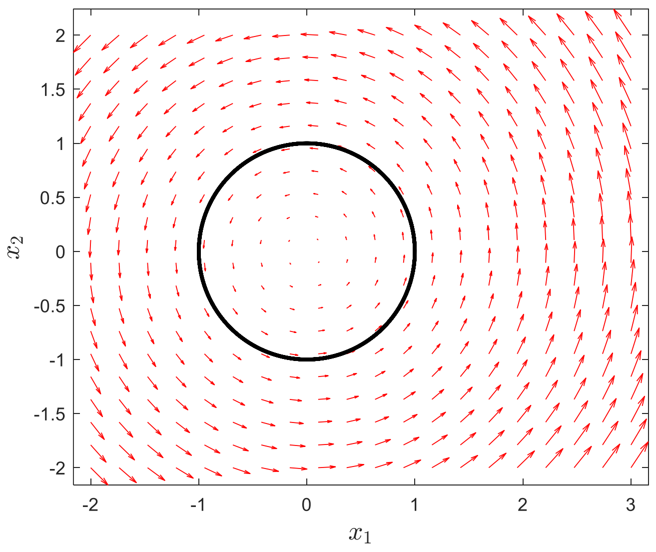
Matching quiz 3

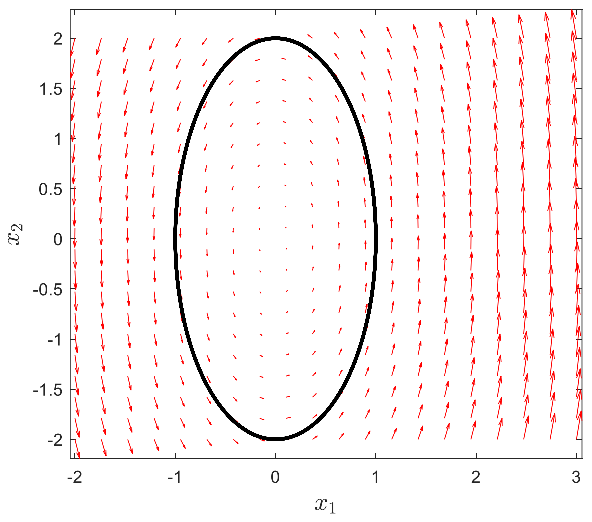
$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} x$$

$$\dot{x} = \begin{bmatrix} -1.5 & -2.5 \\ 2.5 & 1.5 \end{bmatrix} x$$

$$\dot{x} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} x$$







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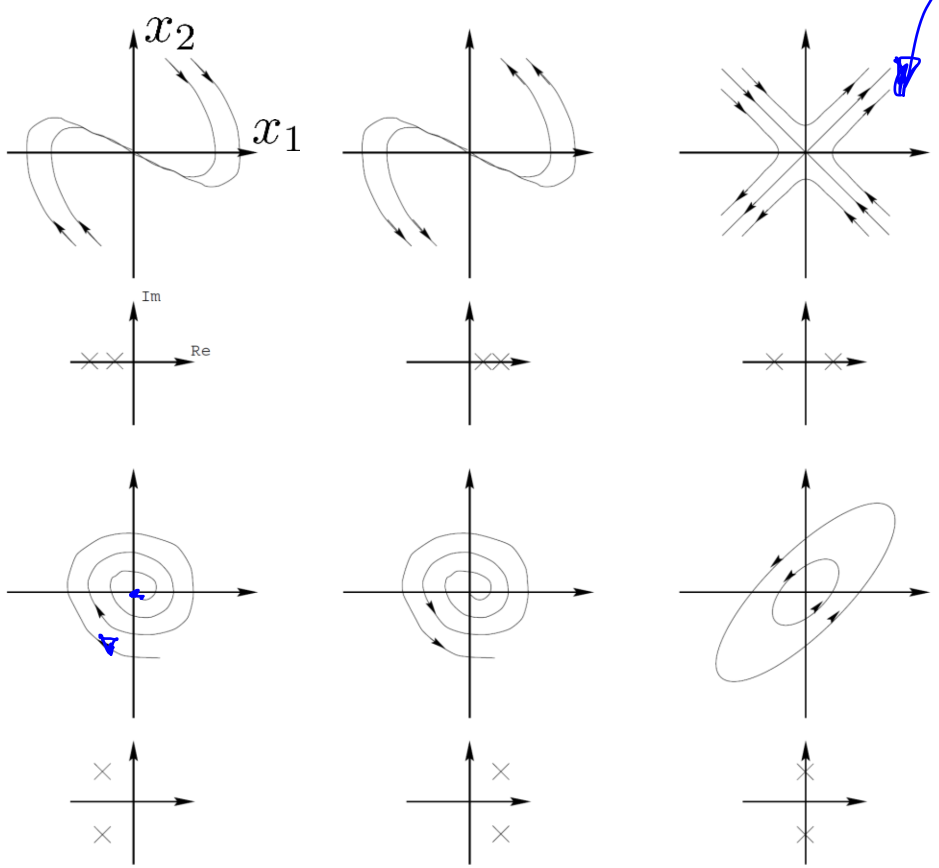
Summary of phase portraits and their equilibriums

Im$\lambda_i = 0$:	stable node $\lambda_1, \lambda_2 < 0$	unstable node $\lambda_1, \lambda_2 > 0$	saddle point $\lambda_1 < 0 < \lambda_2$
Im$\lambda_i \neq 0$:	Re$\lambda_i < 0$ stable focus	Re$\lambda_i > 0$ unstable focus	Re$\lambda_i = 0$ center point



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Effect of perturbations

Perturbations in $A + \Delta$

- Structurally stable: the qualitative behavior remains the same under arbitrarily small perturbations in A

Examples: a node(with distinct eigenvalues), a saddle or a focus

- A stable node with multiple eigenvalues could become a stable node or a stable focus under arbitrarily small perturbations in A

- A center is not structurally stable

$$\dot{z} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} z \quad \Rightarrow \quad \dot{z} = \begin{bmatrix} \delta & -\omega \\ \omega & \delta \end{bmatrix} z \qquad \delta \pm j\omega$$

Back to linearization

Theorem Assume

$$\dot{x} = f(x)$$

is linearized at $x^\star$ so that

$$\dot{\tilde{x}} = A\tilde{x} + g(x),$$

where

- $A = \frac{\partial f}{\partial x}(x^\star)$
- $g = f(x) - \frac{\partial f}{\partial x}(x^\star)\tilde{x} \in C^1$ and $\frac{\|g(x)\|}{\|\tilde{x}\|} \to 0$ as $\|\tilde{x}\| \to 0$

If $\dot{\tilde{x}} = A\tilde{x}$ has a focus, node, or saddle point, then $\dot{x} = f(x)$ has the same type of equilibrium at the origin.

If the linearized system has a center, then the nonlinear system has either a center or a focus.

Back to linearization

Summary

- Linearization

$$\dot{x} = f(x) \Rightarrow \dot{\tilde{x}} = A\tilde{x} + g(x),$$

- Phase portraits of Linear systems
- Whether the behavior of the Linear system (outcome of linearization) can be inherited to the nonlinear system?