Lund University

FRNT05 Nonlinear Control Systems and Servo Systems

Lecture 1: Introduction

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Outline Lecture 1

- Practical information - Course contents
- Nonlinear systems’ representation
- Equilibrium points
- Nonlinear systems vs Linear Systems
- Nonlinear control systems phenomena Phenomena:
  - Finite Escape Time (related to the existence of a solution)
  - Multiple Equilibria
  - Limit Cycles
- Nonlinear differential equations

Course Goal

- To provide students with solid theoretical foundations of nonlinear control systems combined with good engineering ability

- After the course, you should be able to:
  - recognize common nonlinear control problems,
  - use some powerful analysis methods, and
  - use some practical design methods.

Course Material

Textbook(s):

- Glad and Ljung, *Reglerteori, flervariabla och olinjämetoder*, 2003, Studentlitteratur, ISBN 9-14-403003-7 or the English translation *Control Theory*, 2000, Taylor & Francis Ltd, ISBN 0-74-840878-9. The course covers Chapters 11-16,18. (MPC and optimal control not covered in the other alternative textbooks.)

- H. Khalil, Nonlinear Systems (3rd ed.), 2002, Prentice Hall, ISBN 0-13-122740-8. A good, a bit more advanced text.

- ALTERNATIVE: Slotine and Li, Applied Nonlinear Control, Prentice Hall, 1991. The course covers chapters 1-3 and 5, and sections 4.7-4.8, 6.2, 7.1-7.3.

Course Material, cont.

- Handouts (Lecture notes + extra material)
- Exercises (can be downloaded from the course home page)
- Lab PMs 1, 2 and 3
- Home page

http://www.control.lth.se/education

Matlab/Simulink other simulation software (see home page)

Lectures and labs

- The lectures are given in rooms mentioned in TimeEdit as follows:

  - Monday 13–15, weeks 45–49
  - Wednesday 13–15, week 44–49
  - Friday 13-15, week 45–47

- The exercises are held Tuesdays and Thursdays 15:15-17:00 in E:3316.

- The three laboratory experiments are mandatory.
  - Sign-up lists are posted on the web at least one week before the first laboratory session and close one day before.
  - The Laboratory PMs are available at the course homepage.
    Before the lab sessions some home assignments have to be done.
  - No reports after the labs.

The Course

- 14 lectures

- 14 exercises

- 3 laboratories

- 5 hour exam: January 10, 2023, 08:00-13:00, Vic:3C-D.
  Open-book exam: Lecture notes but no old exams/exercises.

Course Outline

- Lecture 1-3 Modelling and basic phenomena

  linearization, phase plane, limit cycles

- Lecture 2-6 Analysis methods

  Lyapunov, circle criterion, describing functions

- Lecture 7-8 Common nonlinearities

  Saturation, friction, backlash, quantization

- Lecture 9-13 Design methods

  Lyapunov methods, Sliding mode & optimal control

- Lecture 14 Summary

State-space models (autonomous systems)

- State vector: x represent the memory that the dynamical system has of its past
- Input vector: u typically control input applied to make the output to behave in a specific manner
- Output vector: y variables of particular interest e.g. measurable or variables required to be controlled

- General: $f(x, u, y, \dot{x}, \dot{u}, \dot{y}, \ldots) = 0$
- Explicit: $\dot{x} = f(x, u), \quad y = h(x)$
- Affine in the control: $\dot{x} = f(x) + g(x)u, \quad y = h(x)$
- Linear time-invariant: $\dot{x} = Ax + Bu, \quad y = Cx$

Affine map

$f(x) = \underbrace{Ax}_{\text{Linear map}} + \underbrace{b}_{\text{"bias"}}$

$\dot{x} = \frac{dx}{dt}$

Non-autonomous systems

- Autonomous forced system: $\dot{x} = f(x, u)$
- Control Input vector: $u = \gamma(t, x)$ may contain functions of time e.g. feedforward terms estimated terms (adaptive control)
- Non-autonomous system: $\dot{x} = f(x, t)$

becomes unforced after substituting explicitely the controller

Always possible to transform to autonomous system

Introduce $x_{n+1} =$ time

$$\begin{aligned} \dot{x} &= f(x, x_{n+1}) \\ \dot{x}_{n+1} &= 1 \end{aligned}$$

if t is a result of adding an integrator in the control input then the system can be transformed to autonomous by introducing $x_{n+1} = \int_0^t x(\tau)d\tau$

$\dot{x}_{n+1} = x$

Linear Systems

State space representation $\dot{x} = \frac{d}{dt}x(t)$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Transfer function

$$H(s) = C(sI - A)^{-1}B + D$$
$$Y(s) = H(s)U(s)$$

State space models of systems are not unique

$$\dot{x} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \ddots & 1 \\ -a_k & -a_{k-1} & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$x = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(k-1)} \end{bmatrix}$$

$$y^{(k)} + a_1 y^{(k-1)} + \cdots + a_{k-1}\dot{y} + a_k y = u$$

- Controllable Canonical Form
- Observable Canonical Form

Linear Systems

State space representation $\dot{x} = \frac{d}{dt}x(t)$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Properties:

Equilibrium $\dot{x} = 0 \Rightarrow Ax = 0$

if A full rank $x = 0$

- Unique equilibrium if A is full-rank
- Regardless of the initial value, the equilibrium point is stable when the eigenvalues have negative real-parts
- Analytic solution: superposition of the natural modes of the system

Lecture 02

Linear Systems

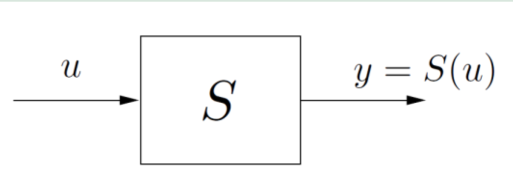
State space representation $\dot{x} = \dfrac{d}{dt}x(t)$

$$Y(s) = H(s)U(s)$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Properties:

- Principle of superposition and scaling



$$S(\alpha u_1 + \beta u_2) = \alpha S(u_1) + \beta S(u_2)$$

- If the unforced system is asymptotically stable, the forced system is bounded-input bounded-output stable
- Sinusoidal input → Sinusoidal output at the same frequency

Properties:

- Unique equilibrium if A is full-rank
- Regardless of the initial value, the equilibrium point is stable when the eigenvalues have negative real-parts
- Analytic solution: superposition of the natural modes of the system

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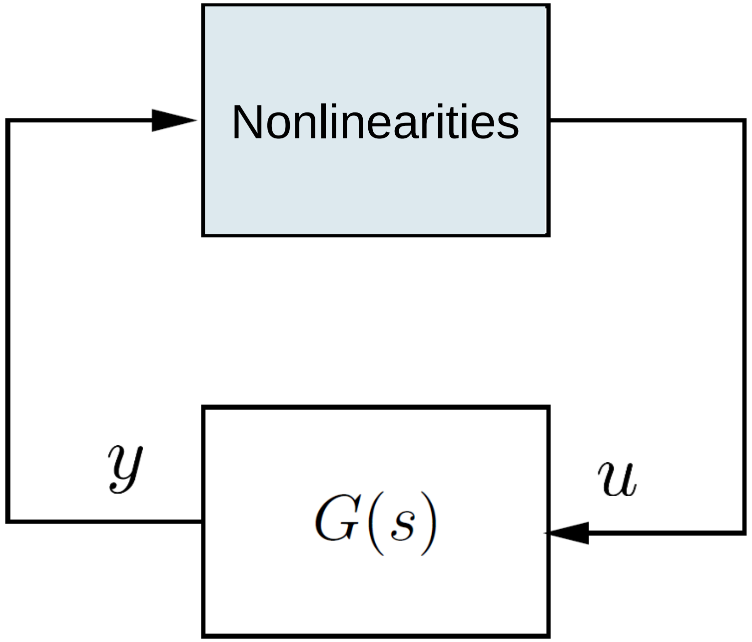
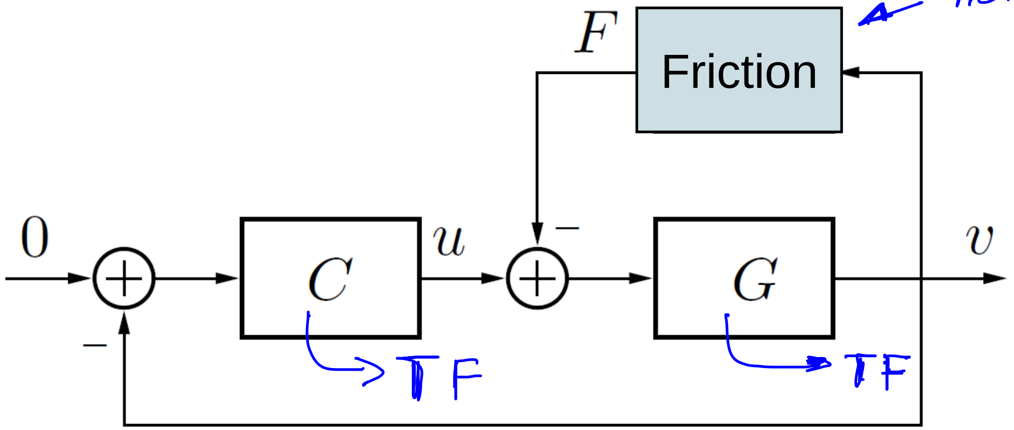
A standard form for analysis



Example: closed loop with friction

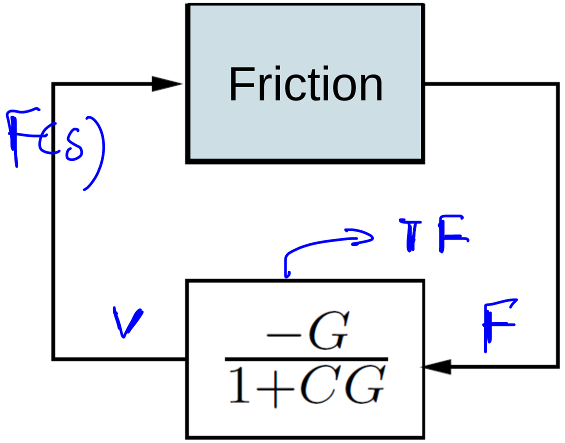


nonlinear function

e.g. $F = -d\, v|v|$

$F = -\mathrm{sgn}(v)$ Coulomb friction

$\Longleftrightarrow$



$$V(s) = G(s)\big(U(s) - F(s)\big) \quad (*)$$

$$U(s) = C(s)\big(0 - V(s)\big) \quad (**)$$

Substituting $(**)$ into $(*)$

$$V(s) = -G(s)C(s)V(s) - G(s)F(s)$$

$$\Rightarrow \big(1 + G(s)C(s)\big)V(s) = -G(s)F(s)$$

$$\Rightarrow (***)$$

$$(***) \quad V(s) = \frac{-G(s)}{1+G(s)C(s)} F(s)$$

First order linear and nonlinear differential equations

- First order unforced systems described by differential equations

nonlinear

$\dot{x} = \frac{dx}{dt}$

$$\underbrace{\dot{x} = -ax}_{\substack{\text{Linear} \\ f(x) = -ax}} \qquad \dot{x} = x^2 \qquad \dot{x} = -\underbrace{|x|}_{\geq 0}x \qquad \dot{x} = x(1-x)$$

- Damping – linear dynamic friction $m\dot{v} + dv = u$

- Damping – nonlinear viscous friction (drag underwater vehicles) $m\dot{v} + d|v|v = u$

- Population growth example $\dot{N} = aN\left(1 - \frac{N}{M}\right)$

N → *Population*

M → *Maximum size that can be reached*

if $N=0$ $\dot{N}=0$ remains 0

if $N = \epsilon > 0$ N will reach the maximum pop. M

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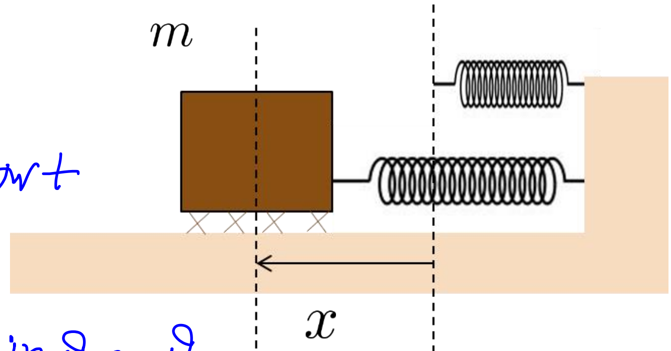
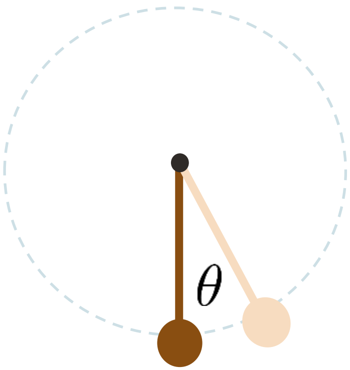
Second order nonlinear equations

$\ddot{x} = \frac{d^2x}{dt^2}$

- Hardening spring $\underbrace{m\ddot{x} + d\dot{x} + k(x}_{\text{linear part}} + \underbrace{ax^3}_{\text{nonlinear part}}) = 0$





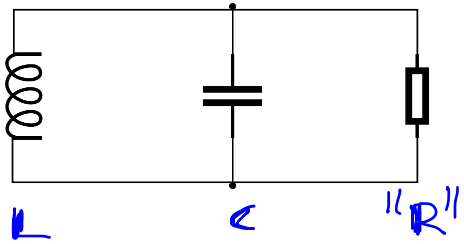
- Pendulum $MR^2\ddot{\theta} + kR\dot{\theta} + MgR\sin\theta = 0$

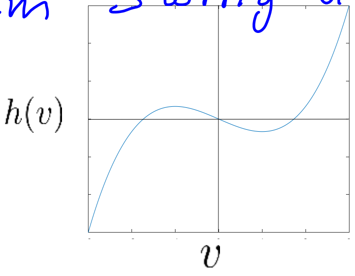
In linear control systems $\sin\theta \sim \theta$ for small angles.

- Mechanical systems with friction $m\ddot{x} + f(x, \dot{x}) = u$

This cannot hold for tasks such as pendulum swing-up

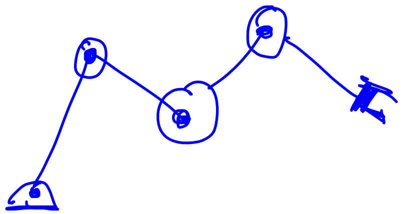
- Circuit with negative resistance $CL\ddot{v} + Lh'(v)\dot{v} + v = 0$





! many electronic components have non-linear characteristics

- Robot manipulators $\underbrace{M\ddot{q}}_{\text{positive definite but nonlinear inertia matrix}} + C(q, \dot{q})\dot{q} + g(q) + F_f(q, \dot{q}) = u$



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Transformation to first order systems & Equilibrium points

$\dot{x} = f(x)$

Assume $y^{(k)} = \frac{d^k y}{dt^k}$ highest derivative of y

Introduce $x = \begin{bmatrix} y & \dot{y} & \dots & y^{(k-1)} \end{bmatrix}^T$

Example: Pendulum

$$MR\ddot{\theta} + k\dot{\theta} + Mg\sin\theta = 0$$

2nd order (DE)

$$x = \begin{bmatrix} \underset{x_1}{\theta} & \underset{x_2}{\dot{\theta}} \end{bmatrix}^T \text{ gives}$$

$$\ddot{\theta} = \frac{1}{MR}\left(-k\underset{x_2}{\dot{\theta}} - Mg\sin\underset{x_1}{\theta}\right)$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{MR}x_2 - \frac{g}{R}\sin x_1 \end{aligned}$$

$\dot{x}_1 = \dot{\theta} = x_2$

$\dot{x}_2 = \ddot{\theta} \overset{DE}{=} -\frac{k}{MR}x_2 - \frac{g}{R}\sin x_1$

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Transformation to first order systems & Equilibrium points

- *The system can stay at equilibrium forever without moving*
- *Set all derivatives equal to zero!*

Assume $y^{(k)} = \frac{d^k y}{dt^k}$ highest derivative of y

Introduce $x = \begin{bmatrix} y & \dot{y} & \dots & y^{(k-1)} \end{bmatrix}^T$

$\Rightarrow$ Set $\dot{x} \equiv \begin{bmatrix} \dot{y} & \ddot{y} & \dots & y^{(k)} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}^T$

General: $f(x_0, u_0, y_0, 0, 0, 0, \dots) = 0$

Explicit: $f(x_0, u_0) = 0$ $\Rightarrow$ nonlinear equation

Linear: $Ax_0 + Bu_0 = 0$ (has analytical solution(s)!) $\Rightarrow$ Linear Systems of Equations

if $u_0 = -Kx_0$ $\Rightarrow$ $(A-BK)x_0 = 0$

if u_0 is given as constant

$Ax_0 = -Bu_0 \Rightarrow x_0 = -A^{-1}Bu_0$

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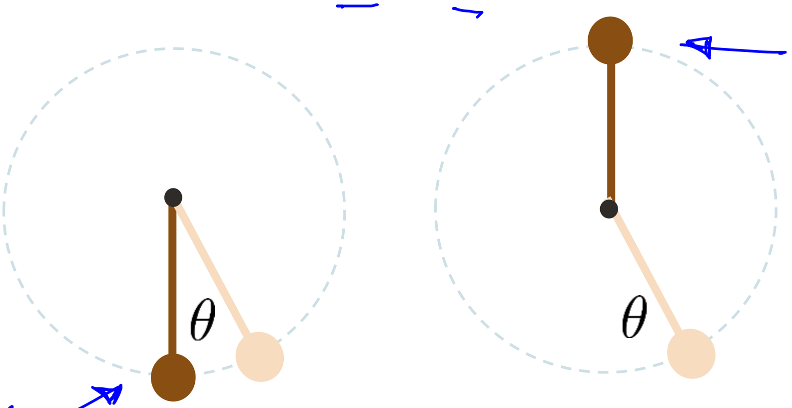
Multiple isolated equilibria

- Pendulum

$$MR\ddot{\theta} + k\dot{\theta} + Mg\sin\theta = 0$$

Equilibria given by $\ddot{\theta} = \dot{\theta} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \quad \theta = \pm n\pi \qquad n \in \{0, 1, 2, 3, ..\}$



unstable (a small disturbance can make it move away from this equilibrium)

stable

- Population growth

$$\dot{N} = aN\left(1 - \frac{N}{M}\right)$$

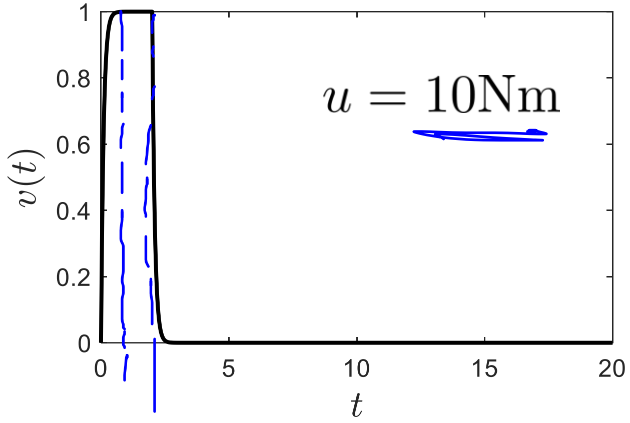
$$N = 0 \qquad N = M$$

Response to the input

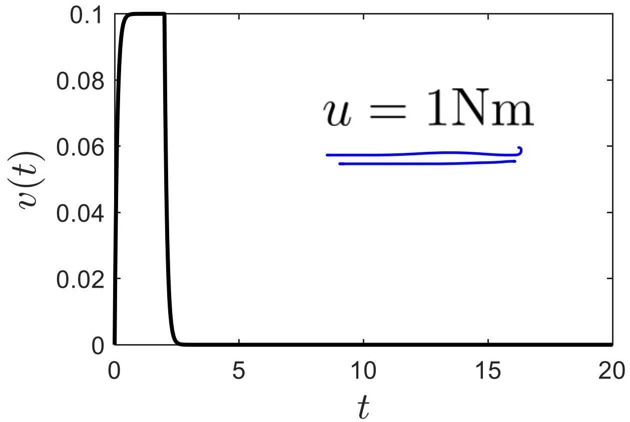
$$m\dot{v} + dv = u$$

pulse input

$m = 1,\ d = 10$

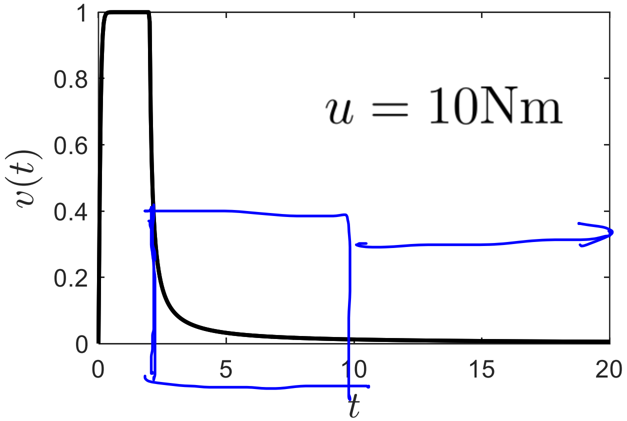


Linear System
Same response
from $0 \to 1$
and $1 \to 0$

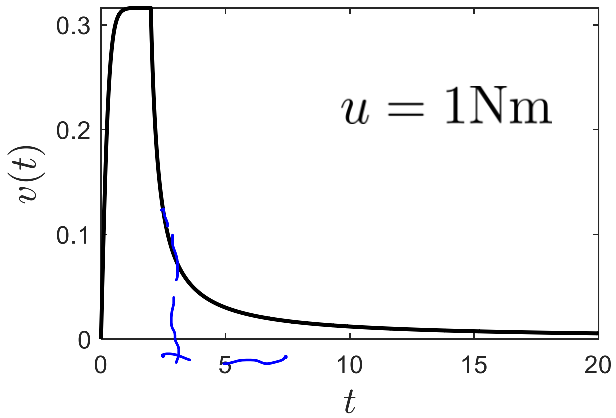


$$m\dot{v} + d|v|v = u$$

$m = 1,\ d = 10$



goes slower
from $1 \to 0$
damping $\sim |v|$
smaller for
low velocities



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Finite escape time

Solution: $\frac{dx}{dt} = x^2 \Rightarrow \frac{dx}{x^2} = dt \Rightarrow \int \frac{dx}{x^2} = dt$

$\Rightarrow -\frac{1}{x} + c = t \Rightarrow \boxed{x = \frac{1}{c-t}} \xRightarrow{c=\frac{1}{x_0}*} x(t) = \frac{x_0}{1-x_0 t}$

* For $t=0$: $-\frac{1}{x_0} + c = 0 \Rightarrow c = \frac{1}{x_0}$

Example: The differential equation

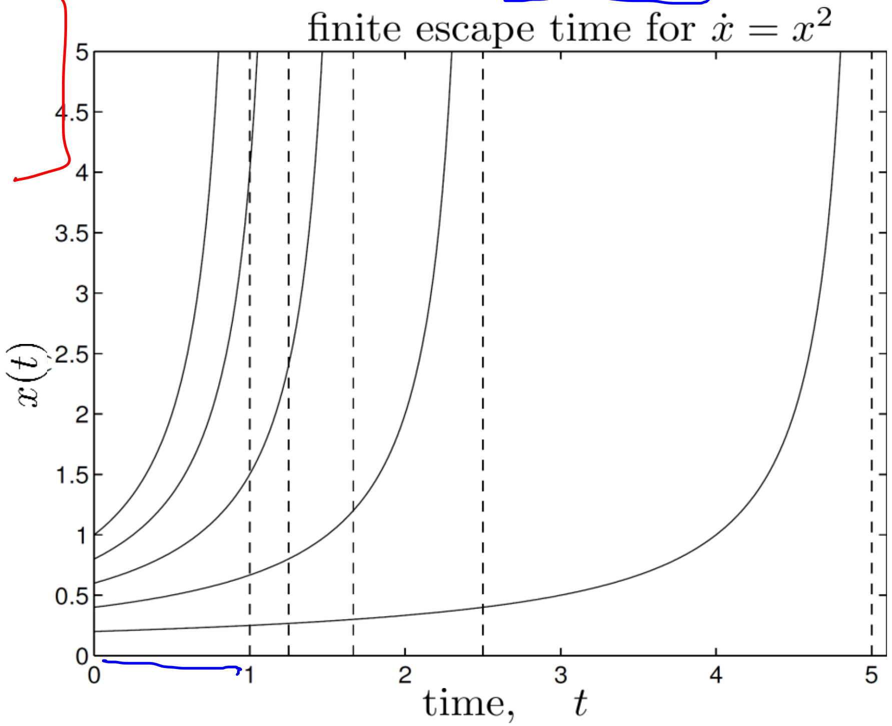
$$\frac{dx}{dt} = x^2, \qquad x(0) = x_0$$

has solution

$$\boxed{x(t) = \frac{x_0}{1 - x_0 t},} \qquad 0 \le t < \frac{1}{x_0}$$

Finite escape time: $t_f = \frac{1}{x_0}$

$$\lim_{t \to t_f} x(t) = \infty$$

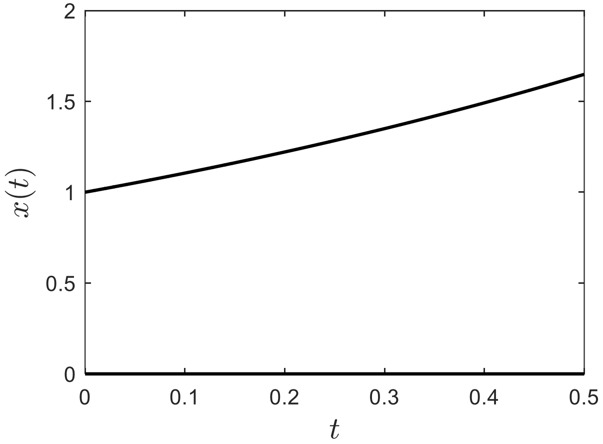


Compare with instability of linear systems

Example: $\dot{x} = +1 x, \quad x(0) = x_0$

Solution: $x(t) = x_0 e^t \qquad \boxed{\lim_{t \to \infty} x(t) = \infty}$

the solution exists $\forall t$ but



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Region of attraction

Region of attraction: The set of all initial conditions such that the solution converges to the equilibrium point

3 equilibrium points

$-x(1-x^2)=0$

$x=0$

$x=+1 \quad x=-1$

Example: The differential equation

$$\dot{x} = -x + x^3, \qquad x(0) = x_0$$

has solution

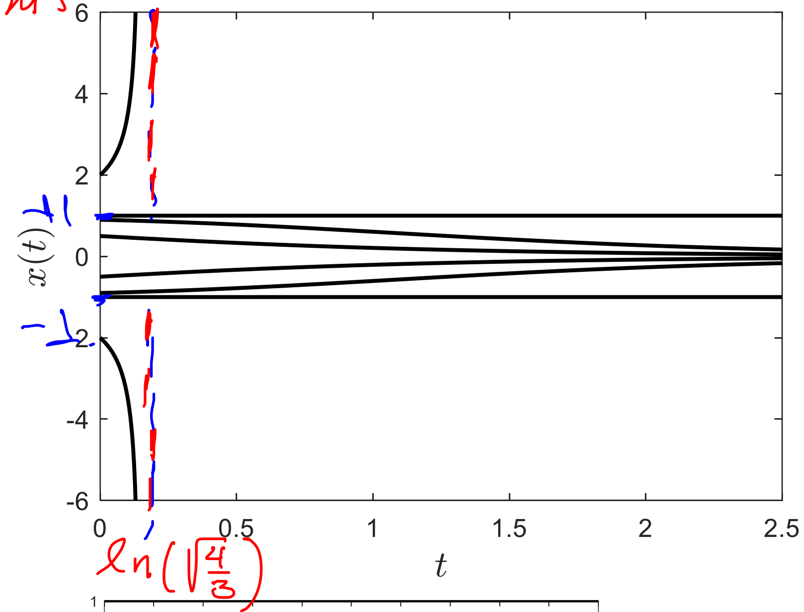
$$x(t) = \frac{x_0 e^{-t}}{\sqrt{1 - x_0^2 + x_0^2 e^{-2t}}}$$

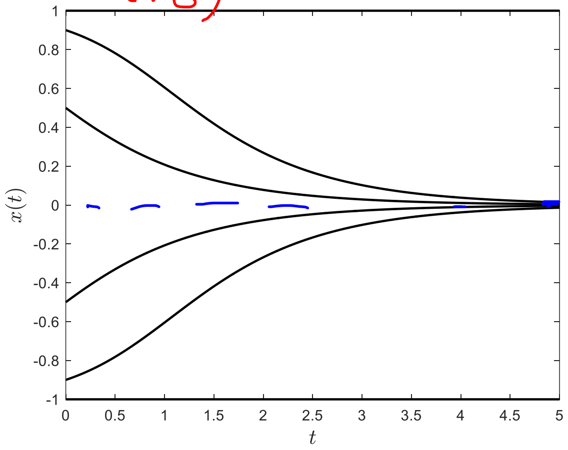
$> 0 \qquad \geq 0$

- If $|x_0| \leq 1$ the solution exists $\forall t \geq 0$
- If $|x_0| > 1$ the solution exists for:

$$0 \leq t < \ln \sqrt{\frac{x_0^2}{x_0^2 - 1}}$$

finite escape time.

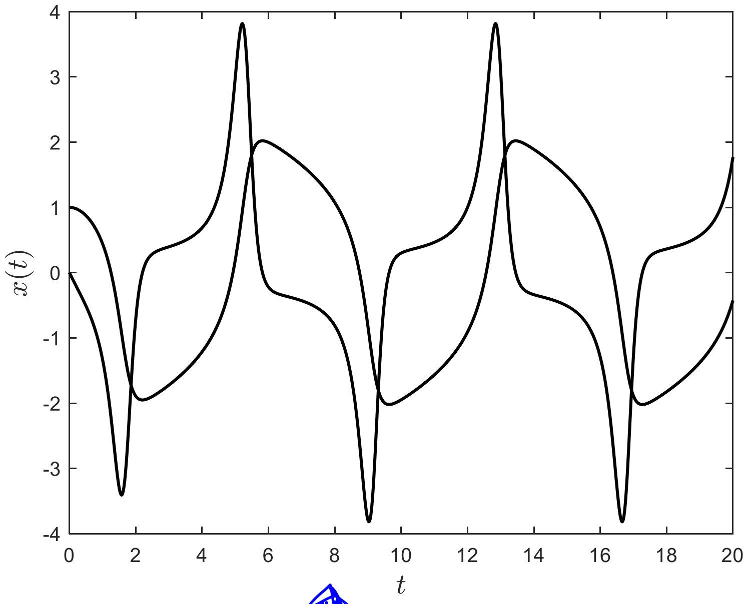


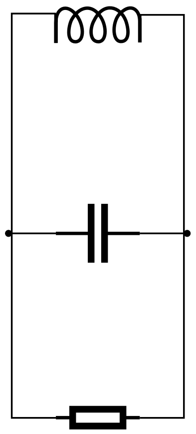


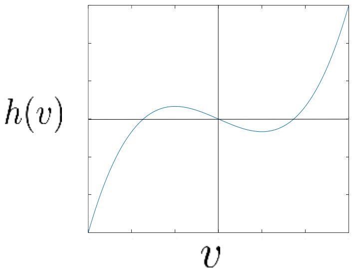
Limit Circles

- Circuit with negative resistance

$$CL\ddot{v} + Lh'(v)\dot{v} + v = 0$$







$$\boxed{\ddot{x} = -2(x^2 - 1)\dot{x} - x}$$

Non linear systems ⇝ limit cycles

Amplitude independent of initial conditions

- Circuit with no resistance

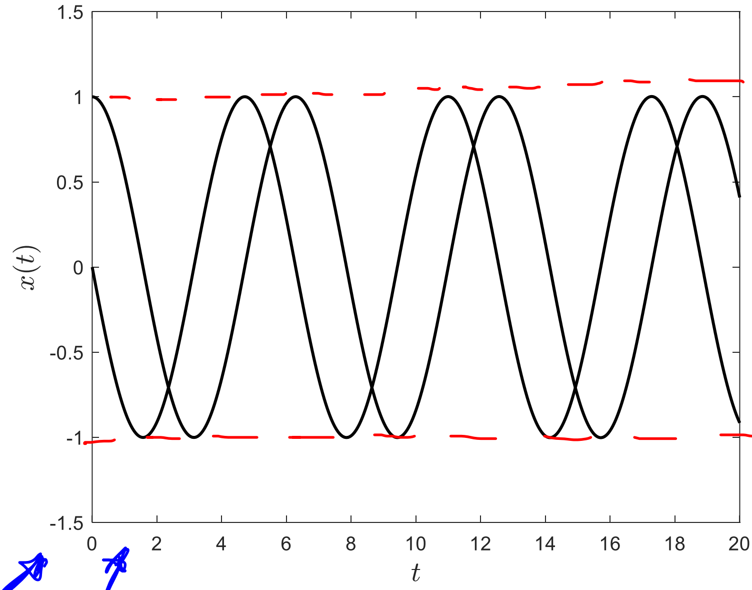
Harmonic Oscillators

Linear Systems

$$CL\ddot{v} + v = 0$$

$v(0) = 1$ $\dot{v}(0) = 0$

$$R = 0$$



The amplitude of the oscillations depends on the initial conditions

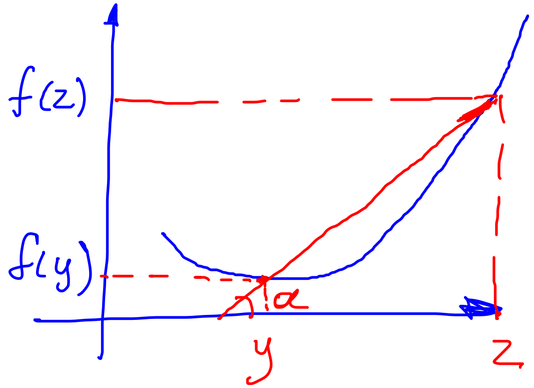
$$\boxed{\ddot{x} = -x}$$

Lipschitz Continuity and existence and uniqueness of solutions

Lipschitz-continuous $\qquad \|f(z)-f(y)\| \leq L\|z-y\|$

Local $\qquad \forall z, y \in \Omega_R$

Global $\qquad \forall z, y \in \Re^n$



A straight line joining any two points of f(x) cannot have a slope greater than L

Infinite slope functions are not locally Lipschitz at the point of discontinuity.

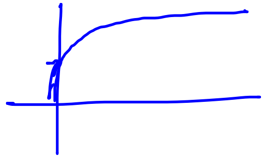
See e.g. $f(x)=\sqrt{x}$

Criteria for checking Lipschitz continuity

Check the continuity of $f(x)$ and the boundedness of first order partial derivatives $\dfrac{\partial f_i}{\partial x_j}$

Examples:

$f(x)=\sqrt{x} \qquad f'(x)=\dfrac{1}{2\sqrt{x}}$



$f(x)=x^3 \qquad f'(x)=3x^2 \rightarrow \infty$ for $x \rightarrow \infty$ locally Lipschitz in a set around 0

$f(x)=\tanh x \qquad f'(x)=1-\tanh^2 x \rightarrow$ globally Lipschitz

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Lipschitz Continuity and existence and uniqueness of solutions

Lipschitz-continuous $\|f(z) - f(y)\| \leq L\|z - y\|$

A solution of the diff. equation $\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = a \end{cases}$ exists

Local $\forall z, y \in \Omega_R$ $\Rightarrow$ $x(t)\,, \quad 0 \leq t < R/C_R,$

$C_R = \max_{x \in \Omega_R} \|f(x)\|$

Global $\forall z, y \in \Re^n$ $\Rightarrow$ $x(t)\,, \quad t \geq 0\,.$

A solution of the diff. equation $\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = a \end{cases}$ exists and is unique if

- $f(x)$ is locally Lipschitz $\forall x \in \Omega_R$
- **and** if it is known that every solution of the differential equation starting at a closed and bounded set $W \subset \Omega_R$ remains in it.

Lipschitz Continuity and existence and uniqueness of solutions

Lipschitz-continuous $\|f(z) - f(y)\| \leq L\|z - y\|$

A solution of the diff. equation $\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = a \end{cases}$ exists

Local $\forall z, y \in \Omega_R$ $\Rightarrow$ $x(t)\,, \quad 0 \leq t < R/C_R,$

$C_R = \max_{x \in \Omega_R} \|f(x)\|$

Global $\forall z, y \in \Re^n$ $\Rightarrow$ $x(t)\,, \quad t \geq 0\,.$

and

Local $\forall z, y \in \Omega_R \qquad x(t) \in W \subset \Omega_R$ $\Rightarrow$ $x(t)\,, \quad t \geq 0\,.$

Check the continuity of $f(x)$ and the boundedness of first order partial derivatives $\dfrac{\partial f_i}{\partial x_j}$

Examples:

$f(x) = \sqrt{x}$ $\qquad f'(x) = \dfrac{1}{2\sqrt{x}}$

$f(x) = x^3$ $\qquad f'(x) = 3x^2$

$f(x) = \tanh x$ $\qquad f'(x) = 1 - \tanh^2 x$

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Lipschitz Continuity and existence and uniqueness of solutions

Lipschitz-continuous	$\|f(z) - f(y)\| \leq L\|z - y\|$	A solution of the diff. equation $\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = a \end{cases}$ exists	
Local	$\forall z, y \in \Omega_R$	$\Rightarrow$	$x(t)\,, \qquad 0 \leq t < R/C_R,$ $\quad C_R = \max_{x \in \Omega_R} \|f(x)\|$
Global	$\forall z, y \in \Re^n$	$\Rightarrow$	$x(t)\,, \qquad t \geq 0\,.$
	and		
Local	$\forall z, y \in \Omega_R \qquad x(t) \in W \subset \Omega_R$	$\Rightarrow$	$x(t)\,, \qquad t \geq 0\,.$

Check the continuity of $f(x)$ and the boundedness of first order partial derivatives $\dfrac{\partial f_i}{\partial x_j}$

Examples:

$$f(x) = \sqrt{x} \qquad f'(x) = \frac{1}{2\sqrt{x}}$$

$$f(x) = x^3 \qquad f'(x) = 3x^2$$

$$f(x) = \tanh x \qquad f'(x) = 1 - \tanh^2 x$$

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Existence of the solution, finite escape time, instability (Quiz)

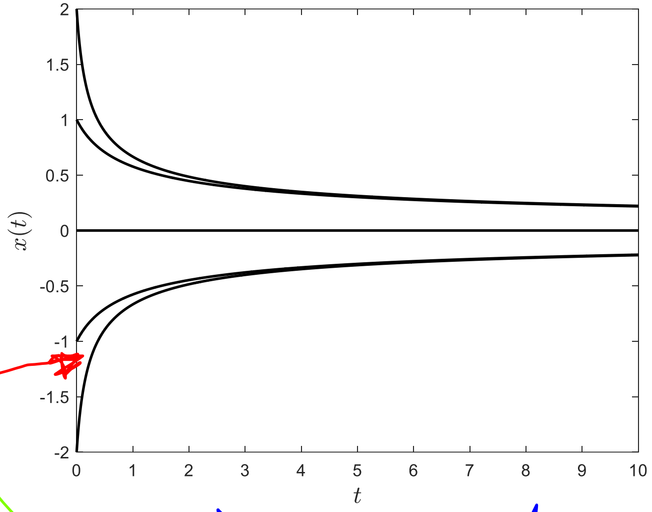
The only system that can have finite escape time

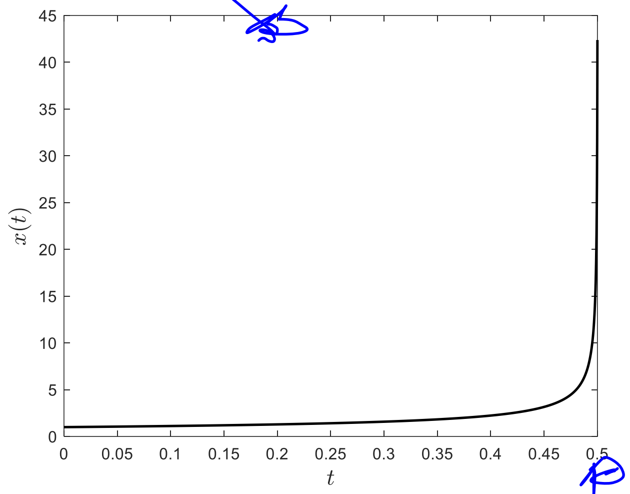
$$\boxed{\dot{x} = x^3}$$

Bounded

$$\dot{x} = -\tanh x$$

globally Lip.





Bounded

$$\dot{x} = -x^3$$

$$= -x^2 \cdot x$$

slower convergence

if $x_0 > 0$ $\dot{x} < 0$

if $x_0 < 0$ $\dot{x} > 0$

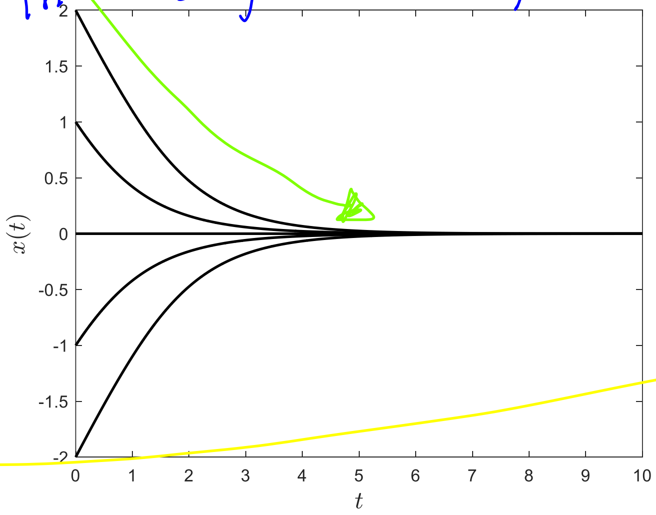
decreasing to zero

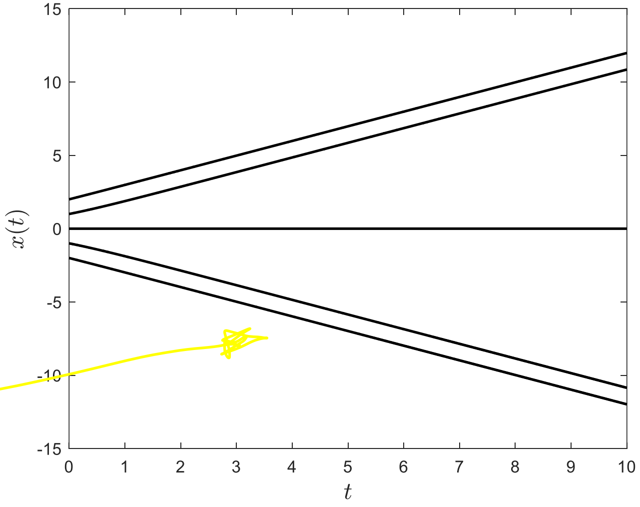
increasing to zero

bounded

globally Lip.

$$\dot{x} = \tanh x$$





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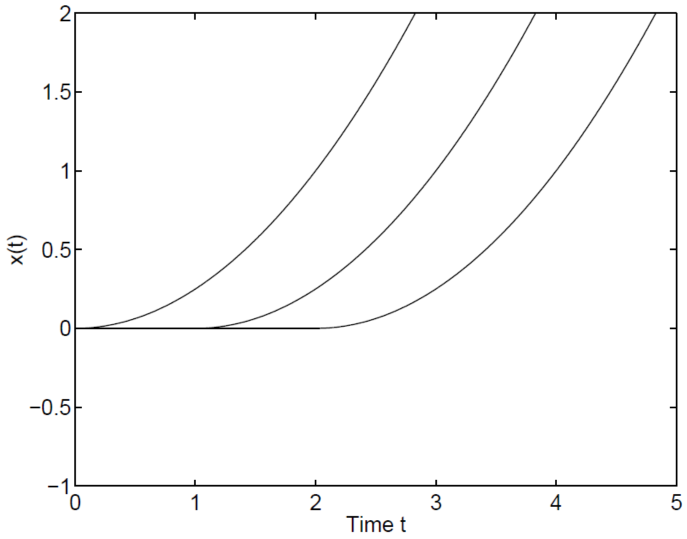
Uniqueness problems

Does the initial value problem have more than one solution?

If so, the differential equation cannot be used for prediction

Example: The equation $\dot{x} = \sqrt{x}, \;\; x(0) = 0$ has many solutions:

$$x(t) \;\; = \;\; \begin{cases} (t-C)^2/4 & t > C \\ 0 & t \le C \end{cases}$$



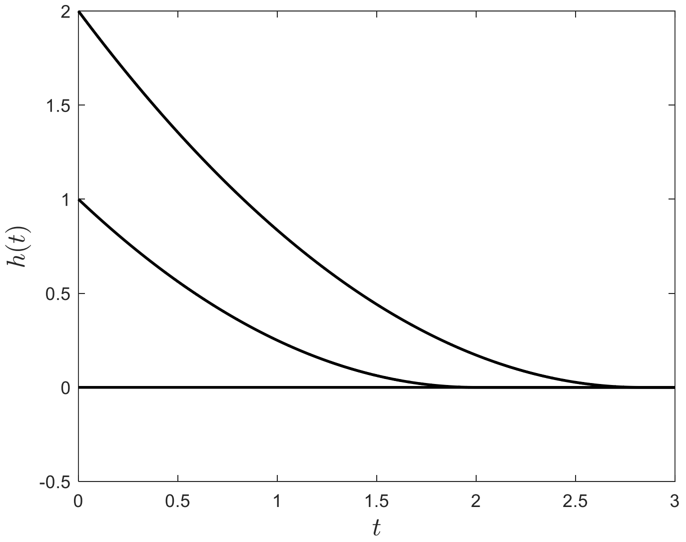
Compare with water tank:

$$\dot{h} = -a\sqrt{h}, \qquad\qquad h : \text{height (water level)}$$

Change to backward-time: “If I see it empty, when was it full?”

$$h(t) \;\; = \;\; \begin{cases} (t-2\sqrt{h_0})^2/4 & t < 2\sqrt{h_0} \\ 0 & t \ge 2\sqrt{h_0} \end{cases}$$

$a = 1, \; h_0 > 0$



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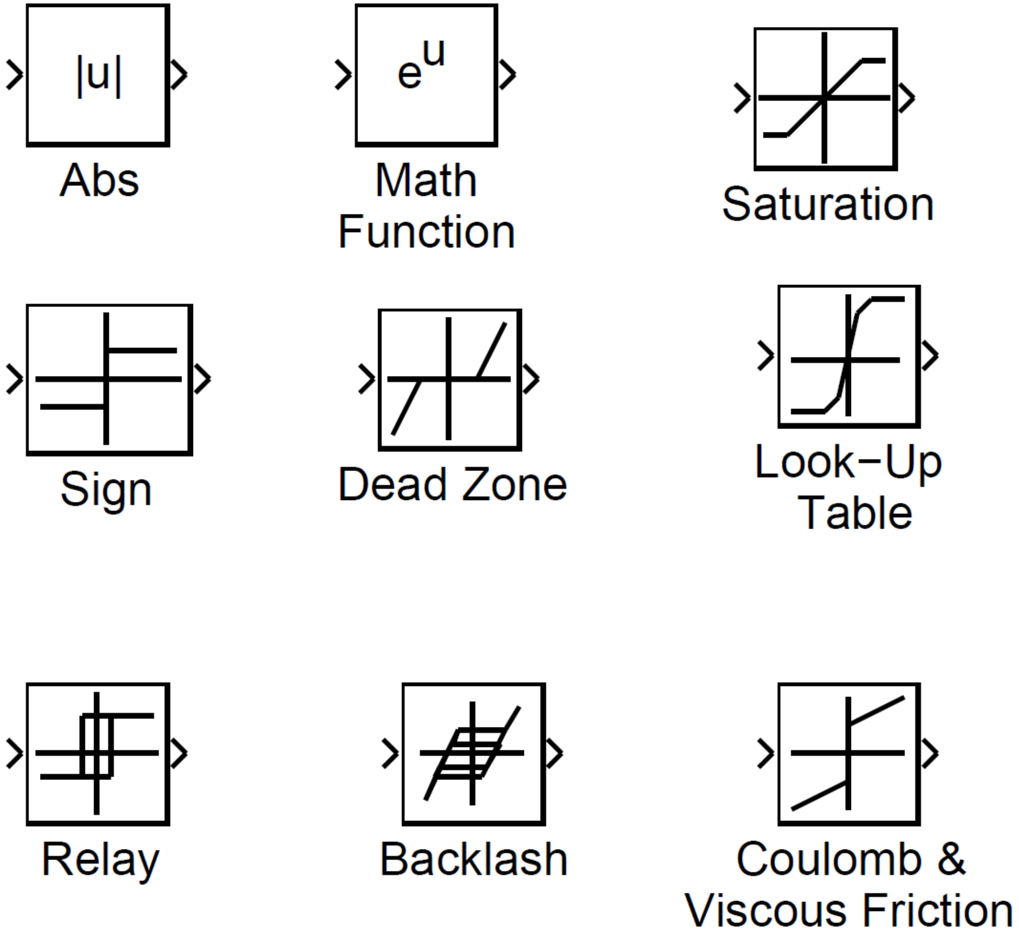
Some nonlinearities -simulink



Next Lecture(s)

- Phase plane analysis for 2nd order linear systems
- Linearization
- Stability definitions
- Simulation in Matlab