[FRTN65] Exercise 12: Identification of linear dynamical systems- Part 2

1 Exercise 1

This exercise has the goal to understand the role of covariance in parametric model estimation.

1. We consider Auto-Regressive (AR) process given by

$$y(t) + a_1 y(t-1) + \cdots + a_n y(t-n) = e(t), \ \mathbb{E}(e^2(t)) = \lambda. \tag{1}$$

   Upon multiplication by $y(t-\tau), \tau \geq 0$ and taking the expectation operator, we arrive at

$$\mathbb{E}[y(t-\tau)y(t)] + a_1\mathbb{E}[y(t-\tau)y(t-1)] + \cdots + a_n\mathbb{E}[y(t-\tau)y(t-n)] = \mathbb{E}[y(t-\tau)e(t)].$$

   This allows to write,

$$R_y(\tau) + a_1 R_y(\tau-1) + \cdots + a_n R_y(\tau-n) = \mathbb{E}[y(t-\tau)e(t)].$$

   Note that for $\tau > 0$, we have $\mathbb{E}[y(t-\tau)e(t)] = 0$, since $e(t)$ is uncorrelated with any past values of $y(t-\tau)$. For $\tau = 0$, we determine $\mathbb{E}[y(t)\, e(t)]$ by multiplying eq. (1) with $e(t)$ and taking the expectation operator, we arrive at $\mathbb{E}[e^2(t)] = \lambda$.

2. We consider the AR- model given by

$$y(t) + a_1 y(t-1) + a_2 y(t-2) = e(t), \ \mathbb{E}(e^2(t)) = \lambda. \tag{2}$$

   By using the result from a), we can write

$$R_y(\tau) + a_1 R_y(\tau-1) + a_2 R_y(\tau-2) = \begin{cases} \lambda & \tau = 0 \\ 0 & \tau > 0 \end{cases}$$

   - For $\tau = 0$,
$$R_y(0) + a_1 R_y(1) + a_2 R_y(2) = \lambda.$$
   - For $\tau = 1$,
$$R_y(1) + a_1 R_y(0) + a_2 R_y(1) = 0.$$
   - For $\tau = 2$,
$$R_y(2) + a_1 R_y(1) + a_2 R_y(0) = 0.$$

   In vector form, we can write

$$\begin{bmatrix} 1 & a_1 & a_2 \\ a_1 & 1+a_2 & 0 \\ a_2 & a_1 & 1 \end{bmatrix} \begin{bmatrix} R_y(0) \\ R_y(1) \\ R_y(2) \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix}. \tag{3}$$

3. There are *two* ways of estimating models in Matlab: either hand-coded (where for example using least square estimate is calculated by entering the solution) or through default function pertaining to the identification toolbox in Matlab. Example of these function are: `idpoly, getcov, present`. Using the Matlab `help` function or online documentation, their functionalities can be checked.

2 Exercise 2

The goal of this exercise is to compare ARX model described by

$$A(q)\,y(t) = B(q)\,u(t) + e(t), \tag{4}$$

with OE model described by,

$$y(t) = \frac{B(q)}{F(q)}u(t) + e(t). \tag{5}$$

We make the following remarks:

- For the AR- process, `arx3` and `arx15` are good models for high-frequencies (above the cross-over frequency), where `arx9` is better suitable for low-frequencies. As can be seen, there is no ideal ARX- model that captures the frequency behavior over all frequencies. This can also be read from the time evolution of the measured output `ytest` and `ypred50w`.
- For the AR- process, the default estimated model `oe` behaves poorly in particular for low-frequencies. If we use the weighting filter using $\cdot$ command and input the particular frequency range of interest, i.e. $[0, 10]$ in this case, we obtain `oe3w`, which shows drastic improvement in the low frequency estimate.

3 Exercise 3

In this exercise, we understand the role of the weighting filter.

- Based on `prbs` input, we can excite the system with different frequencies (corresponding to different periods M). The higher is frequency $1/M$, the more jumps has the resulting prbs signal.
- The filtered signals y_0, y_1, y_2, y_3 show the effect of the third-order filter Y_k, in the decrease in the slope of the amplitude magnitude and the decrease of phase angles.
- Using the interactive tool, it is possible to choose a model that best fits the data and this can be presented into Matlab console.

4 Exercise 4

Given two independent signals $x(t), y(s)$ for all $t, s > 0$, with $\mathbb{E}(x(t)) = 0$, $\mathbb{E}(x^2(t)) = R_x$ and $\mathbb{E}(y(t)) = 0$ and $\mathbb{E}(y^2(t)) = R_y$, our goal in this exercise is to calculate the variance of

$$\hat{R}_{xy}(\tau) = \frac{1}{N}\sum_{t=1}^{N} x(t+\tau)\,y(t). \tag{6}$$

We have

$$\mathrm{Var}[\hat{R}_{xy}(\tau)] = \mathbb{E}[\hat{R}^2_{xy}(\tau)] - \underbrace{\mathbb{E}[\hat{R}_{xy}(\tau)]^2}_{0},$$

since $\frac{1}{N}\sum_{t=1}^{N} \mathbb{E}[x(t+\tau)\,y(t)] = 0$. Hence

$$\begin{aligned}
\mathrm{Var}[\hat{R}_{xy}(\tau)] &= \mathbb{E}[\hat{R}^2_{xy}(\tau)] \\
&= \frac{1}{N^2}\mathbb{E}\left[\left(\sum_{t=1}^{N} x(t+\tau)\,y(t)\right)\left(\sum_{t=1}^{N} x(t+\tau)\,y(t)\right)\right] \\
&= \frac{1}{N^2}\mathbb{E}[\underbrace{x(1)x(N)y(1)y(N)}_{R_x(N-1)R_y(N-1)} + \underbrace{x(2)x(N)y(2)y(N) + x(1)x(N-1)y(1)y(N-1)}_{2R_x(N-2)R_y(N-2)} + ... \\
&+ \underbrace{x(1)x(1)y(1)y(1) + x(2)x(2)y(2)y(2) + ... + x(N)x(N)y(N)y(N)}_{NR_x(0)R_y(0)} + ... \\
&+ \underbrace{x(N)x(1)y(N)y(1)}_{R_x(1-N)R_y(1-N)}], \\
&= \frac{1}{N}\sum_{\tau=-(N-1)}^{N-1} \frac{N-|\tau|}{N} R_x(\tau)R_y(\tau).
\end{aligned}$$

Here we have re-sorted the N^2 different terms and used the fact that $\mathbb{E}[x(t_1)x(t_2)y(t_3)y(t_4)] = \mathbb{E}[x(t_1)x(t_2)]\,\mathbb{E}[y(t_3)y(t_4)]$ when x and y are independent signals. For large N if we approximate $\frac{N-|\tau|}{N} = 1$ we can conclude that

$$\mathrm{Var}[\hat{R}_{xy}(\tau)] \approx \frac{1}{N}\sum_{\tau=-\infty}^{\infty} R_x(\tau)R_y(\tau).$$

(To justify the last step theoretically would require some further conditions on the behavior of $R_x(\tau)$ and $R_y(\tau)$ for large $|\tau|$).