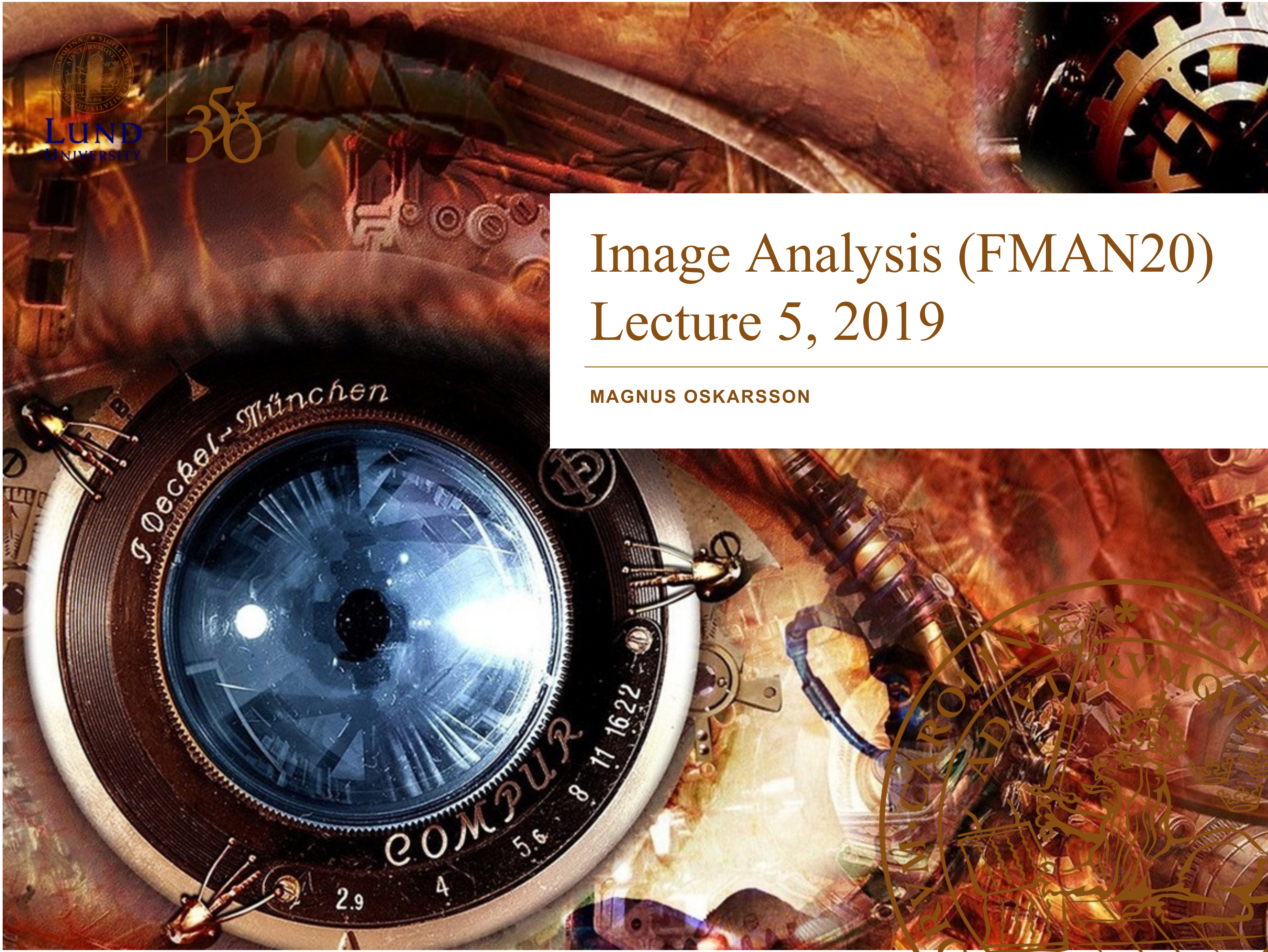


Image Analysis (FMAN20)
Lecture 5, 2019

MAGNUS OSKARSSON

LUND UNIVERSITY

Image Analysis - Motivation

Overview – Machine Learning 1

1. Machine Learning
2. Bayes Theorem
   1. Counting
   2. Binning
   3. Curse of dimensionality
   4. Adaptive binning (K-means)
3. Nearest Neighbour, K-NN
4. Parametric Density Estimation (Plug-in classifier)
5. Logistic regression

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

Machine Learning – Bayes rule

Assume that one feature vector $\mathbf{x}$ and class y are drawn from a joint probability distribution. If one can calculate the probability that the class is $y = j$ given the measurements $\mathbf{x}$, i.e. the so called **posterior probability**.

$$P(y = j|\mathbf{x})$$

The **maximum a posteriori classifier** is obtained as selecting the class j that maximizes the posterior probability, i.e.

$$j = \text{argmax}_k \, P(y = k|\mathbf{x}).$$

It is often easier to model and estimate the **likelihood** $P(\mathbf{x}|y = j)$ and to model the **prior** $p(y = j)$. The a posteriori probabilites can then be calculated using the Bayes rule,

$$p(y = j|\mathbf{x}) = \frac{p(\mathbf{x}|y = j)p(y = j)}{p(\mathbf{x})}.$$

Example

- In a small town, there are two bicycle brands **albatross** and **butterfly**. Albatross sell mostly **green** bikes (80 percent) are green and the rest are **yellow**. Butterfly sell 50 percent green bikes and 50 percent yellow. The Albatross brand is more popular. They have 90 percent of the market in the town. **In another nearby town**, from a distance you see a yellow bike. What is the probability that the bike is an Butterfly bike?

Joint probabilites P(x,y)

- Joint probability and the prior $P(\mathbf{x}) = \sum_{\mathbf{y}} P(\mathbf{x}, \mathbf{y})$

- Joint probability and the total probability $P(\mathbf{y}) = \sum_{\mathbf{x}} P(\mathbf{x}, \mathbf{y})$

$$P(\mathbf{y}) = \sum_{\mathbf{x}} P(\mathbf{x}, \mathbf{y})$$

P(x,y)	x=Green	x=Yellow		
y=Albatross	0,72	0,18	0,9	p(y=Albatross)
y=Butterfly	0,05	0,05	0,1	p(y=Butterfly)
	0,77	0,23	1	
	p(x=Green)	p(x=Yellow)		

P(x|y)	x=Green	x=Yellow	
y=Albatross	0,80	0,20	1,00
y=Butterfly	0,50	0,50	1,00
	1,30	0,70	2,00

$$P(\mathbf{x}) = \sum_{\mathbf{y}} P(\mathbf{x}, \mathbf{y})$$

A posteriori probabilites

- A posteriori probabilites

$$P(\mathbf{y}|\mathbf{x}) = P(\mathbf{x}, \mathbf{y})/P(\mathbf{x})$$

P(x,y)	x=Green	x=Yellow		
y=Albatross	0,72	0,18	0,9	p(y=Albatross)
y=Butterfly	0,05	0,05	0,1	p(y=Butterfly)
	0,77	0,23	1	
	p(x=Green)	p(x=Yellow)		

P(x|y)	x=Green	x=Yellow	
y=Albatross	0,80	0,20	1,00
y=Butterfly	0,50	0,50	1,00
	1,30	0,70	2,00

P(y|x)	x=Green	x=Yellow	
y=Albatross	0,94	0,78	1,72
y=Butterfly	0,06	0,22	0,28
	1	1	2

Usual workflow

- Model prior $P(\mathbf{y})$ and measurement probabilites $P(\mathbf{x}|\mathbf{y})$
- Joint probability $P(\mathbf{x},\mathbf{y}) = P(\mathbf{x}|\mathbf{y})P(\mathbf{y})$
- Total probability $P(\mathbf{x}) = \sum_{\mathbf{y}} P(\mathbf{x},\mathbf{y})$
- A posteriori probability $P(\mathbf{y}|\mathbf{x}) = P(\mathbf{x},\mathbf{y})/P(\mathbf{x})$
- Classify according to maximum a posteriori probability

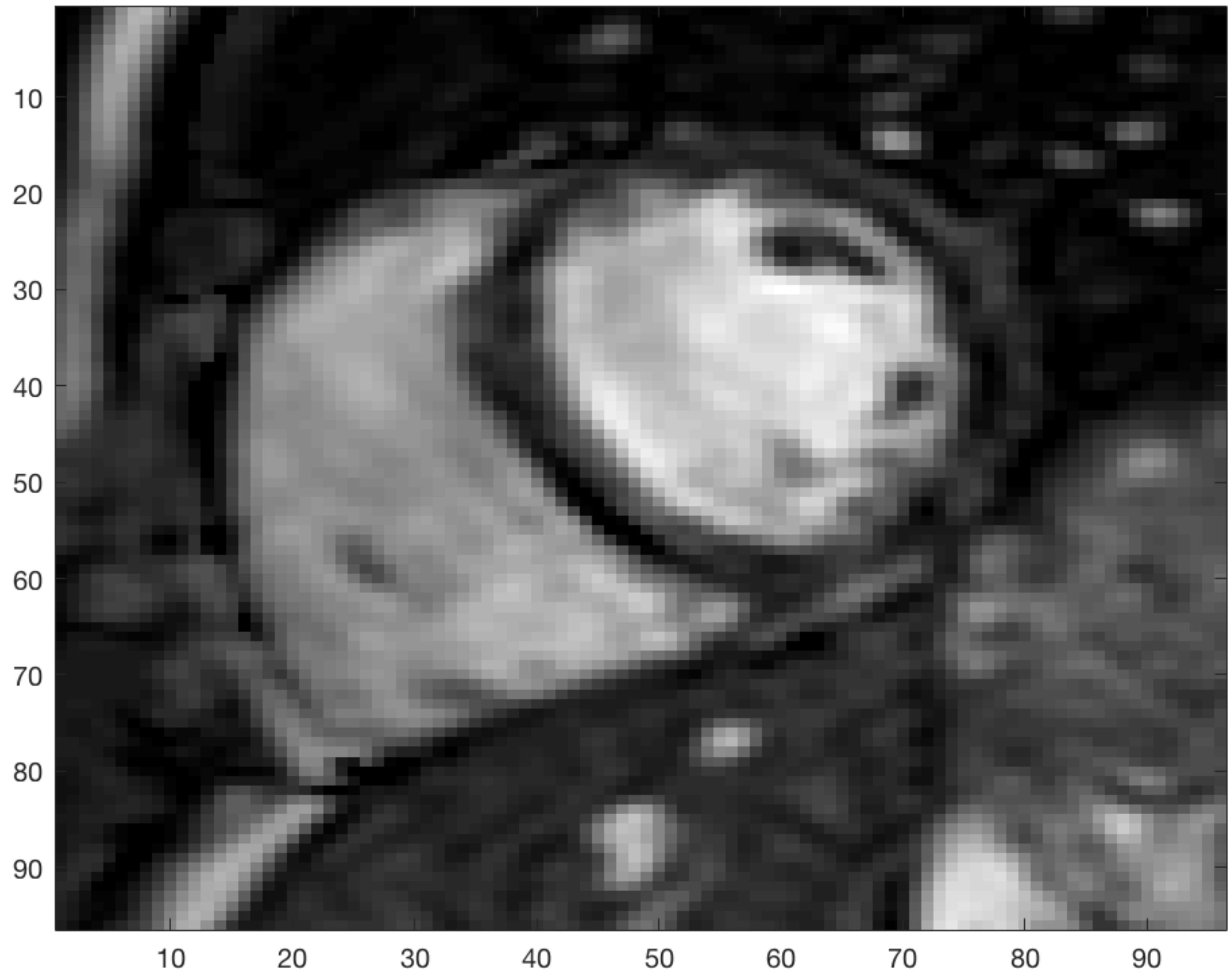
P(x,y)	x=Green	x=Yellow		
y=Albatross	0,72	0,18	0,9	p(y=Albatross)
y=Butterfly	0,05	0,05	0,1	p(y=Butterfly)
	0,77	0,23	1	
	p(x=Green)	p(x=Yellow)		

P(x|y)	x=Green	x=Yellow	
y=Albatross	0,80	0,20	1,00
y=Butterfly	0,50	0,50	1,00
	1,30	0,70	2,00

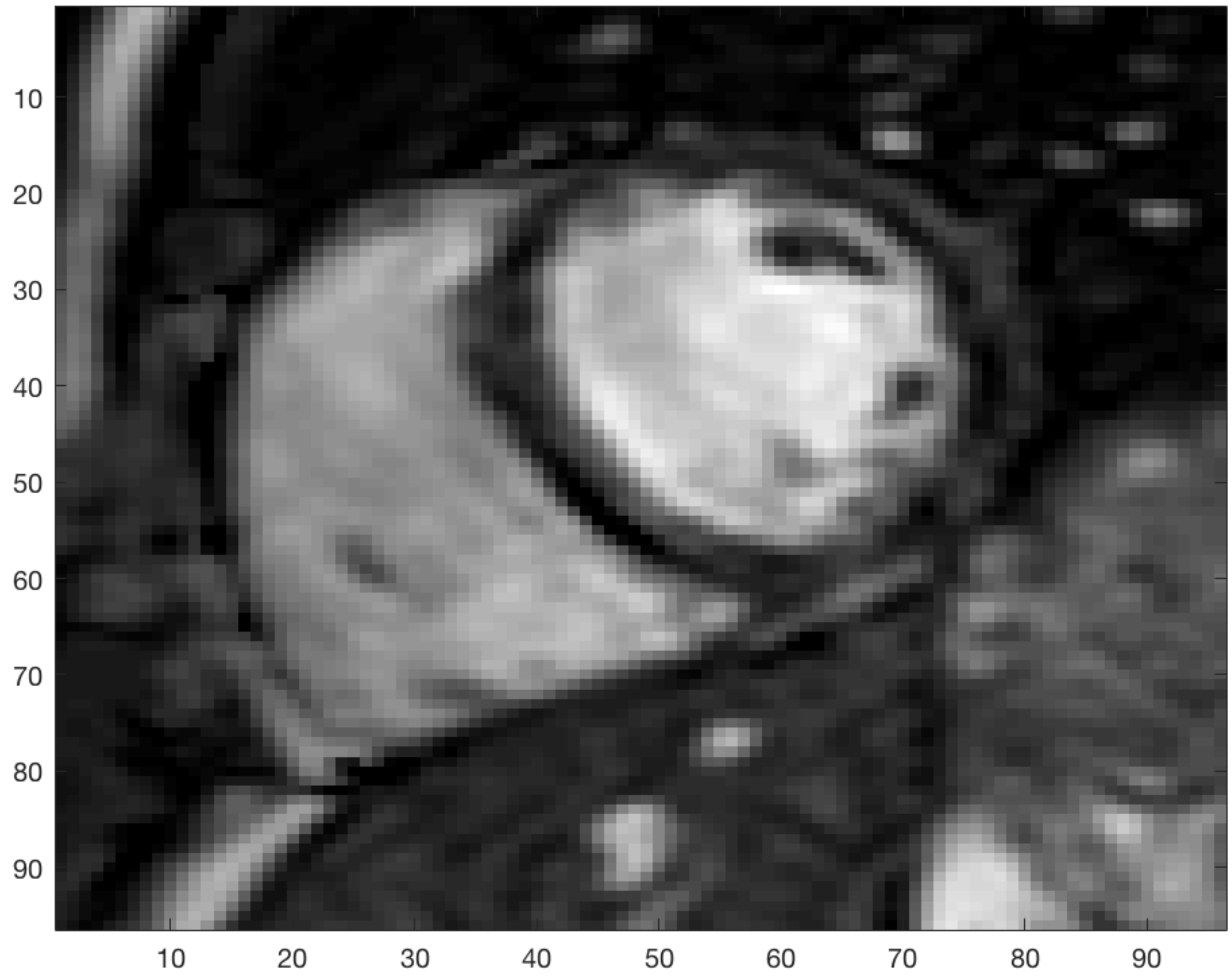
P(y|x)	x=Green	x=Yellow	
y=Albatross	0,94	0,78	1,72
y=Butterfly	0,06	0,22	0,28
	1	1	2

Another example – heart pixels

(We assume that we have segmented and annotated a number of heart images)



Use binning (quantization) to get fewer gray levels. Here 6 levels



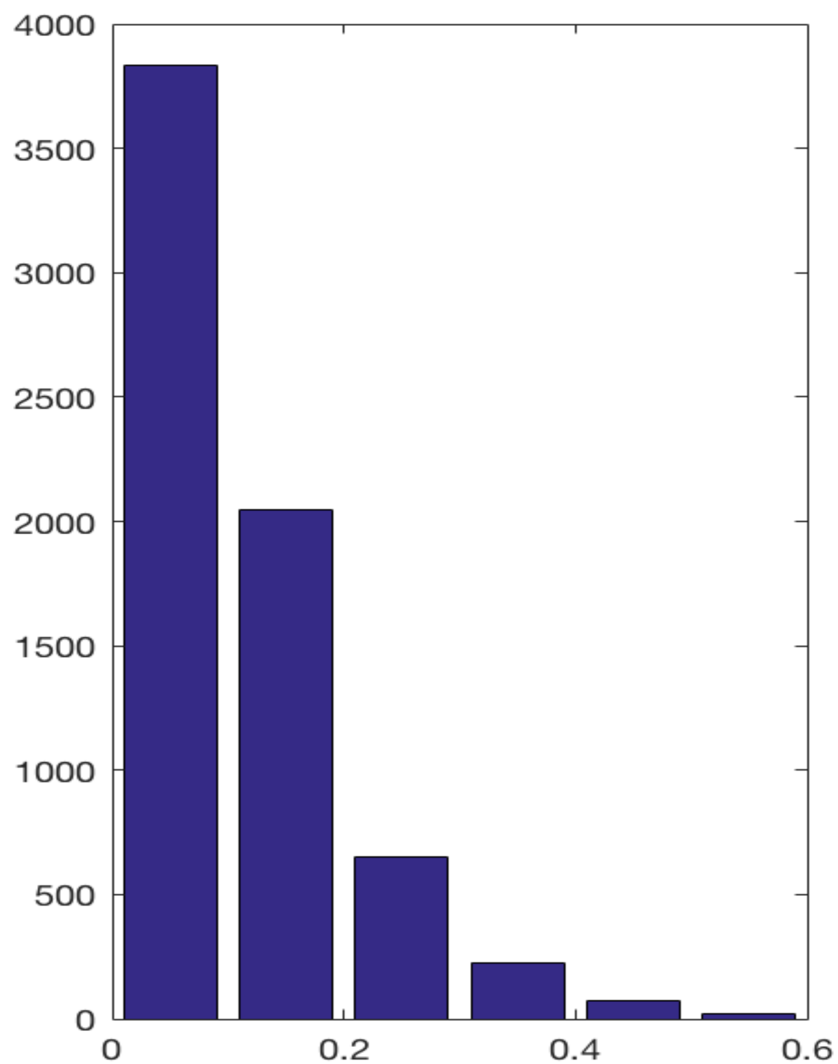
Discretize pixel brightness using 6 bins. Estimate probabilites from training data

P(x | background) P(x | heart)

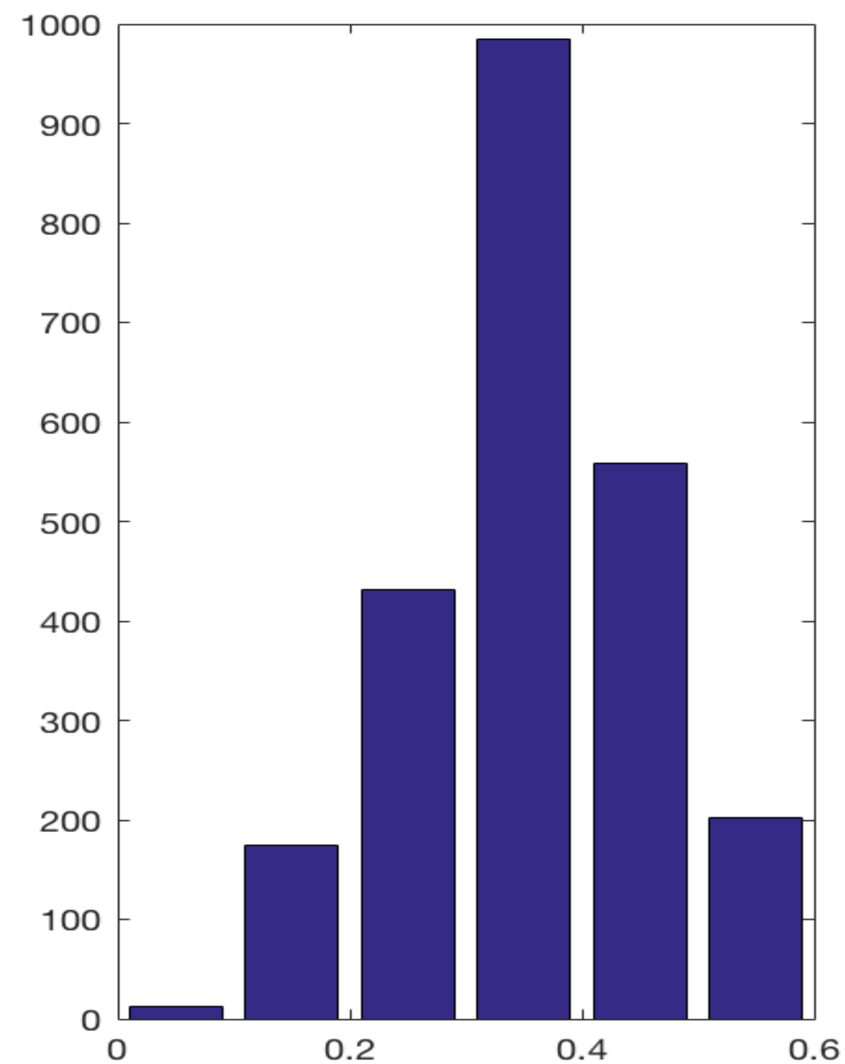
P(x|y)	x=1	x=2	x=3	x=4	x=5	x=6	
y=heart	0,0051	0,0737	0,1825	0,417	0,2363	0,0855	1,00
y=backgroun	0,56	0,30	0,095	0,03	0,01	0,00	1,00
	0,56	0,37	0,28	0,45	0,25	0,09	2,00

Discretize pixel brightness using 6 bins. Estimate probabilites from training data

P(x | background)



P(x | heart)



Estimate a posteriori probabilites.

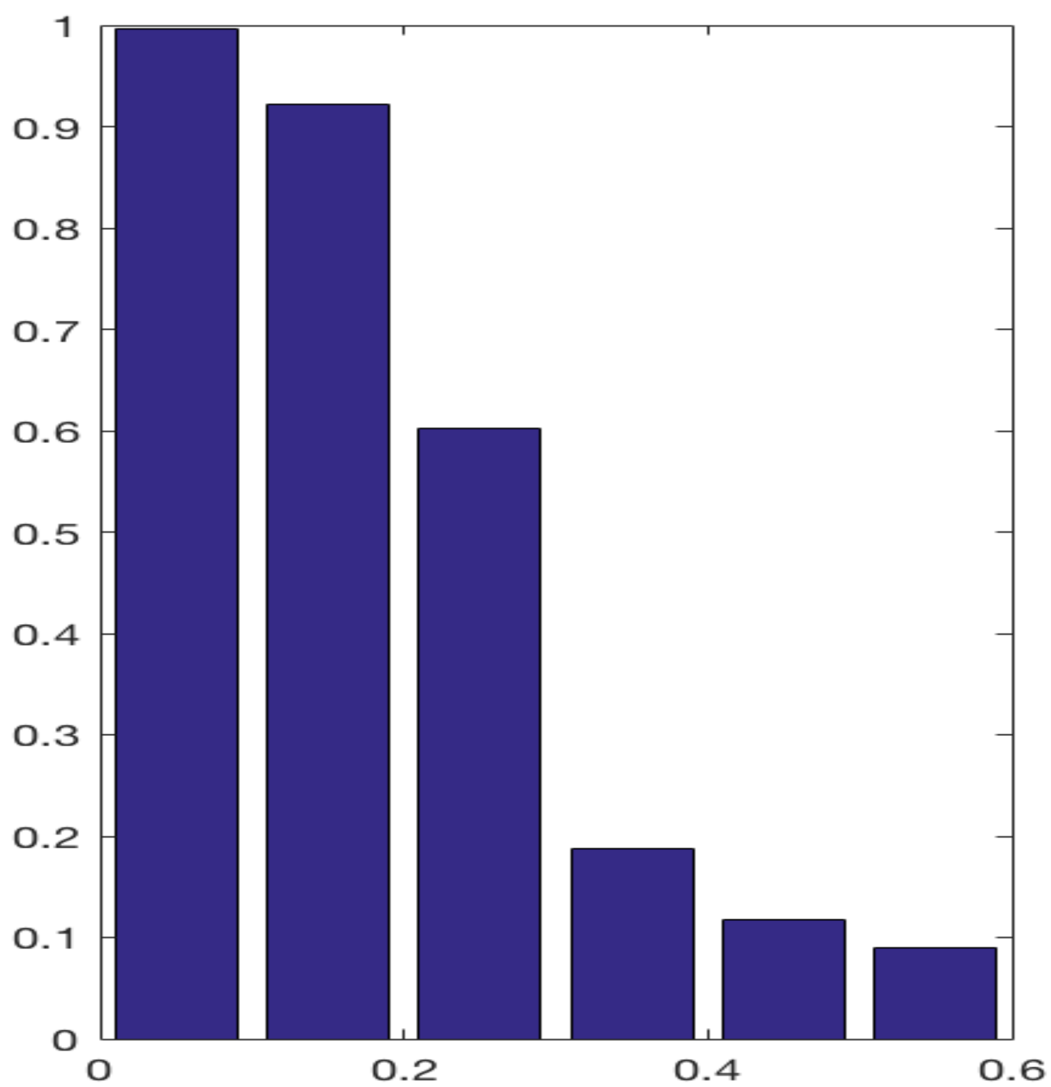
P(x|y)	x=1	x=2	x=3	x=4	x=5	x=6	
y=heart	0,0051	0,0737	0,1825	0,417	0,2363	0,0855	1,00
y=backgroun	0,56	0,30	0,095	0,03	0,01	0,00	1,00
	0,56	0,37	0,28	0,45	0,25	0,09	2,00

P(x,y)	x=1	x=2	x=3	x=4	x=5	x=6		
y=heart	0,001326	0,019162	0,04745	0,10842	0,061438	0,02223	0,26	p(y=heart)
y=backgroun	0,413956	0,221112	0,0703	0,024494	0,007992	0,002146	0,74	p(y=background)
	0,415282	0,240274	0,11775	0,132914	0,06943	0,024376	1	
	p(x=1)	p(x=2)	p(x=3)	p(x=4)	p(x=5)	p(x=6)		

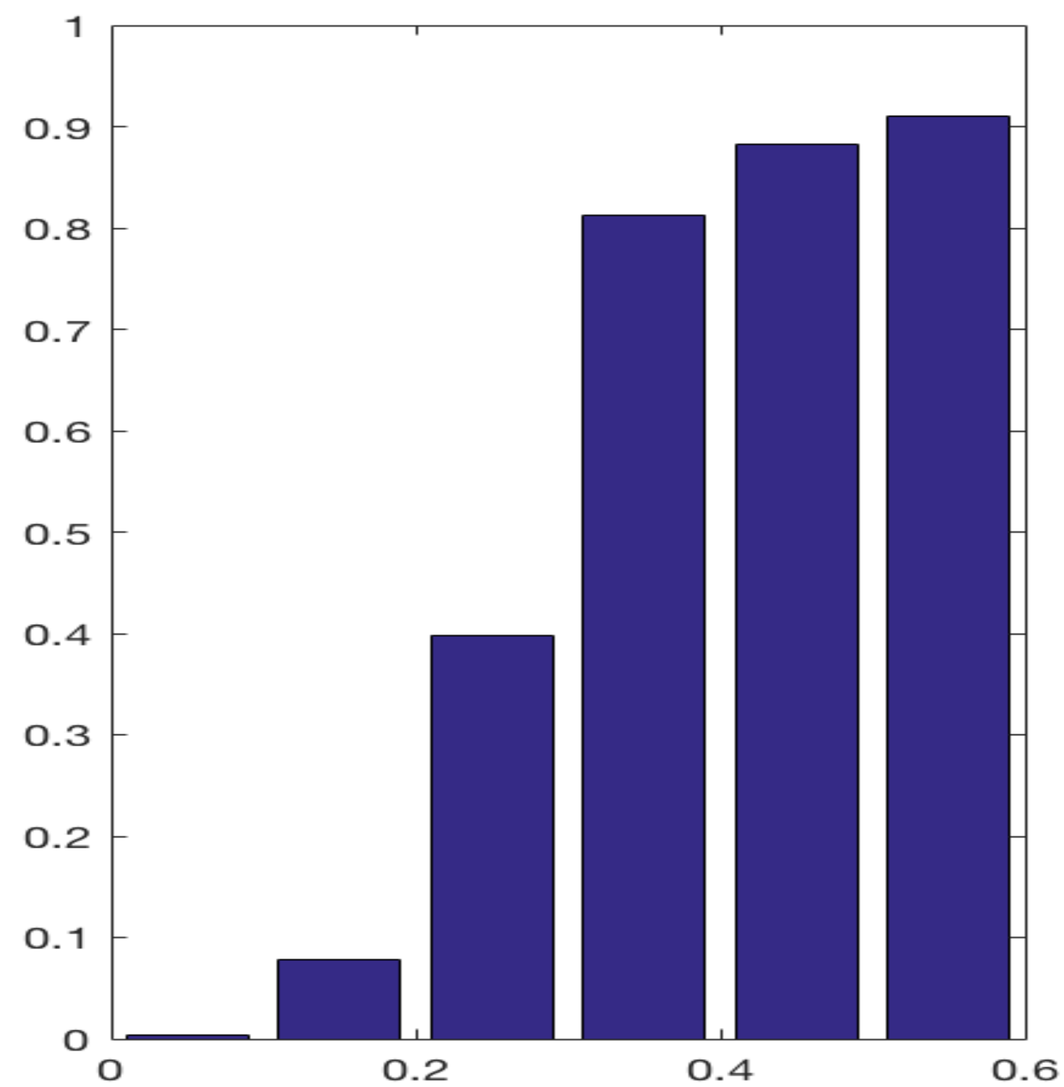
P(y|x)	x=1	x=2	x=3	x=4	x=5	x=6	
y=heart	0,00	0,08	0,40	0,82	0,88	0,91	0,92
y=backgroun	1,00	0,92	0,60	0,18	0,12	0,09	1,08
	1	1	1	1	1	1	2

Estimate a posteriori probabilites. Use these as gray level transforms

P(background | x)



P(heart | x)

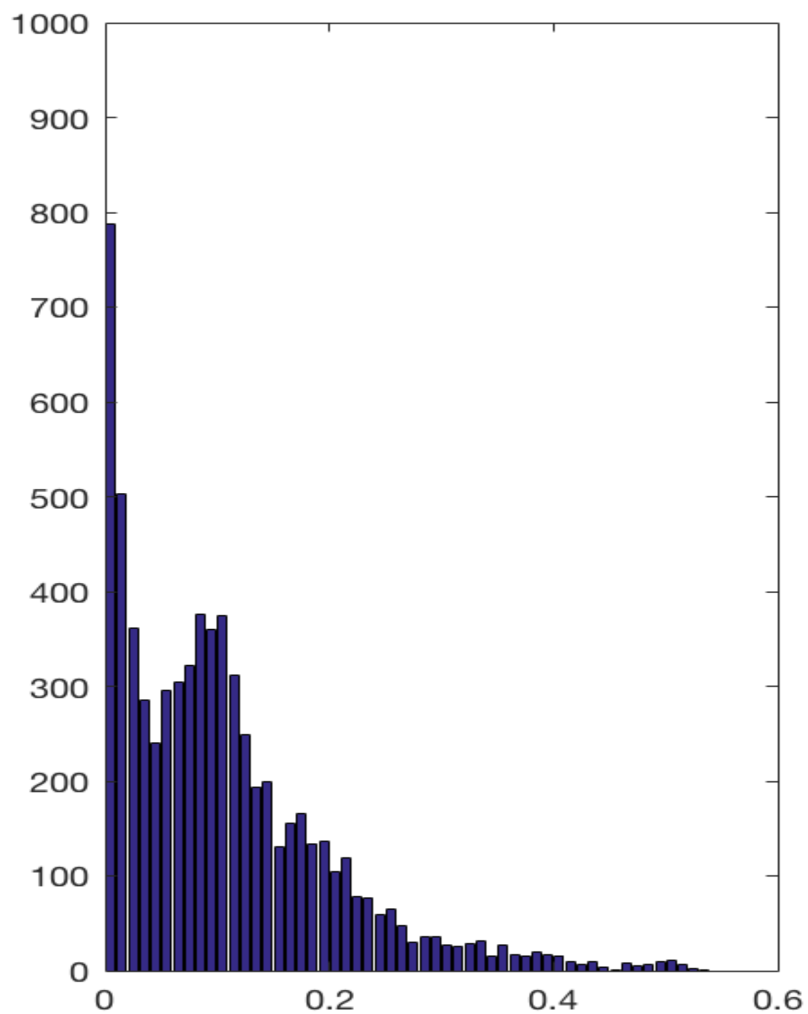


Discretize pixel brightness using 6 bins. Estimate probabilites

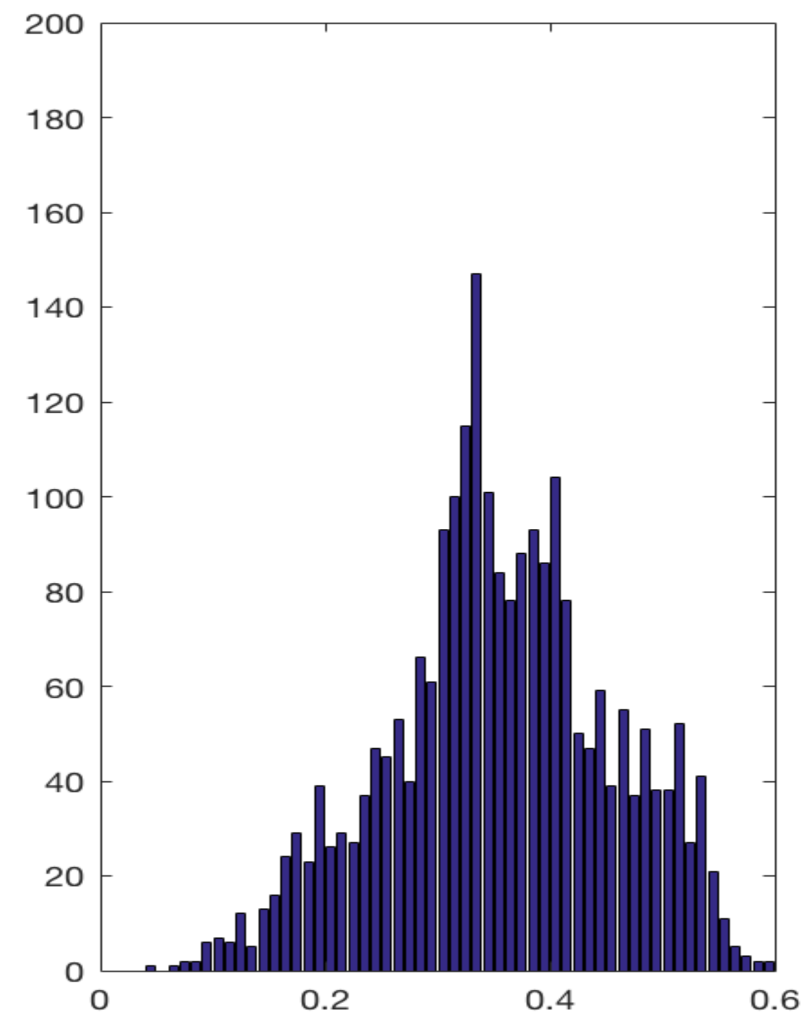
P(heart | x)

Discretize pixel brightness using 60 bins. Estimate probabilites

P(x | background)

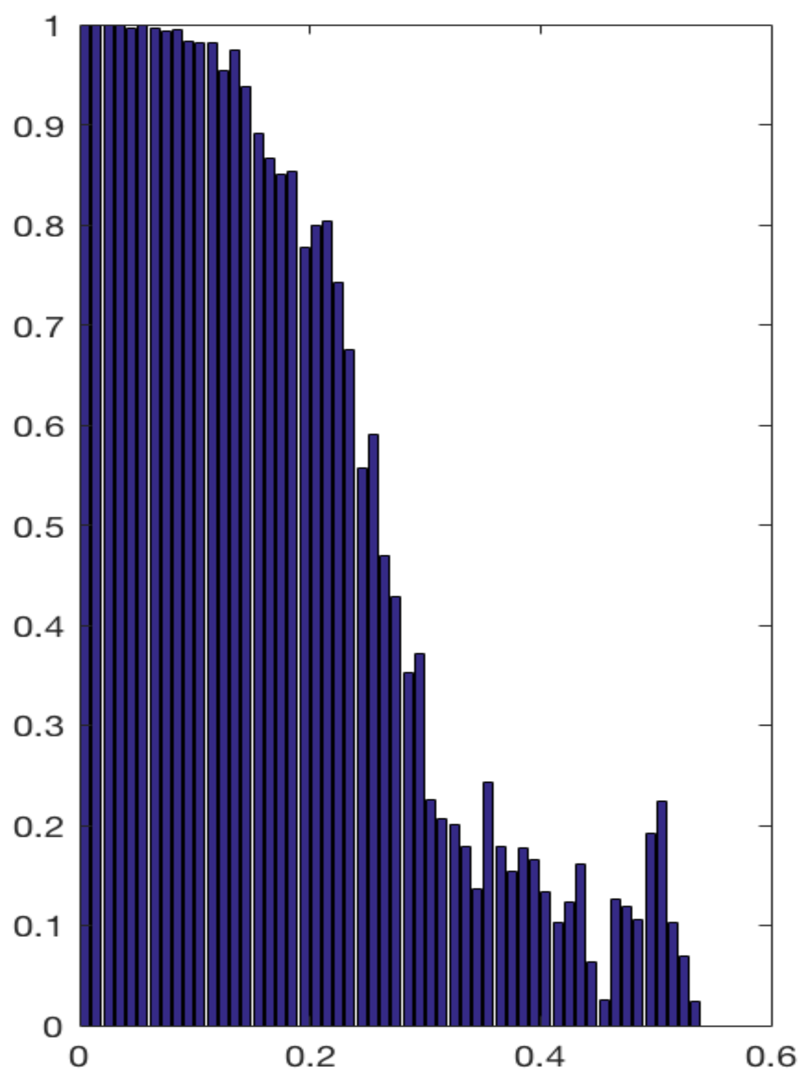


P(x | heart)

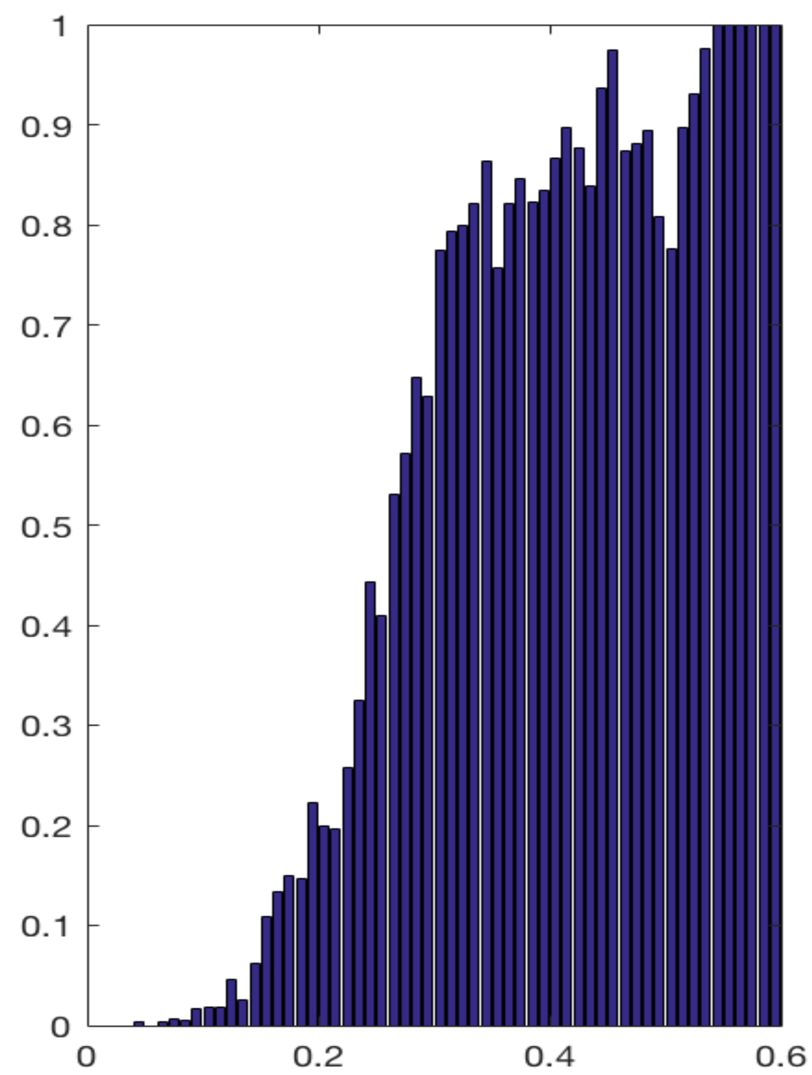


Estimate a posteriori probabilites.
Use these as gray level transforms

P(background | x)

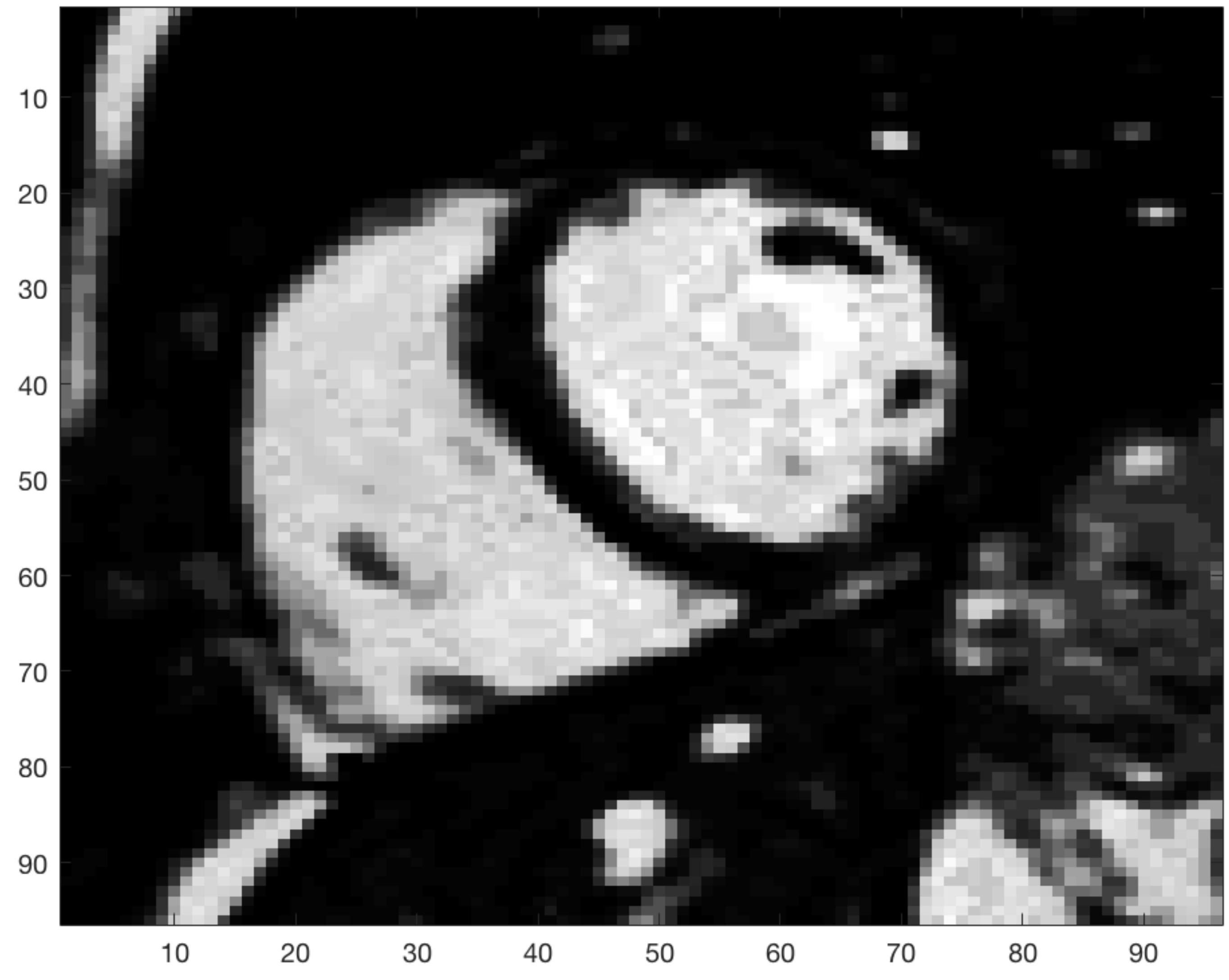


P(heart | x)



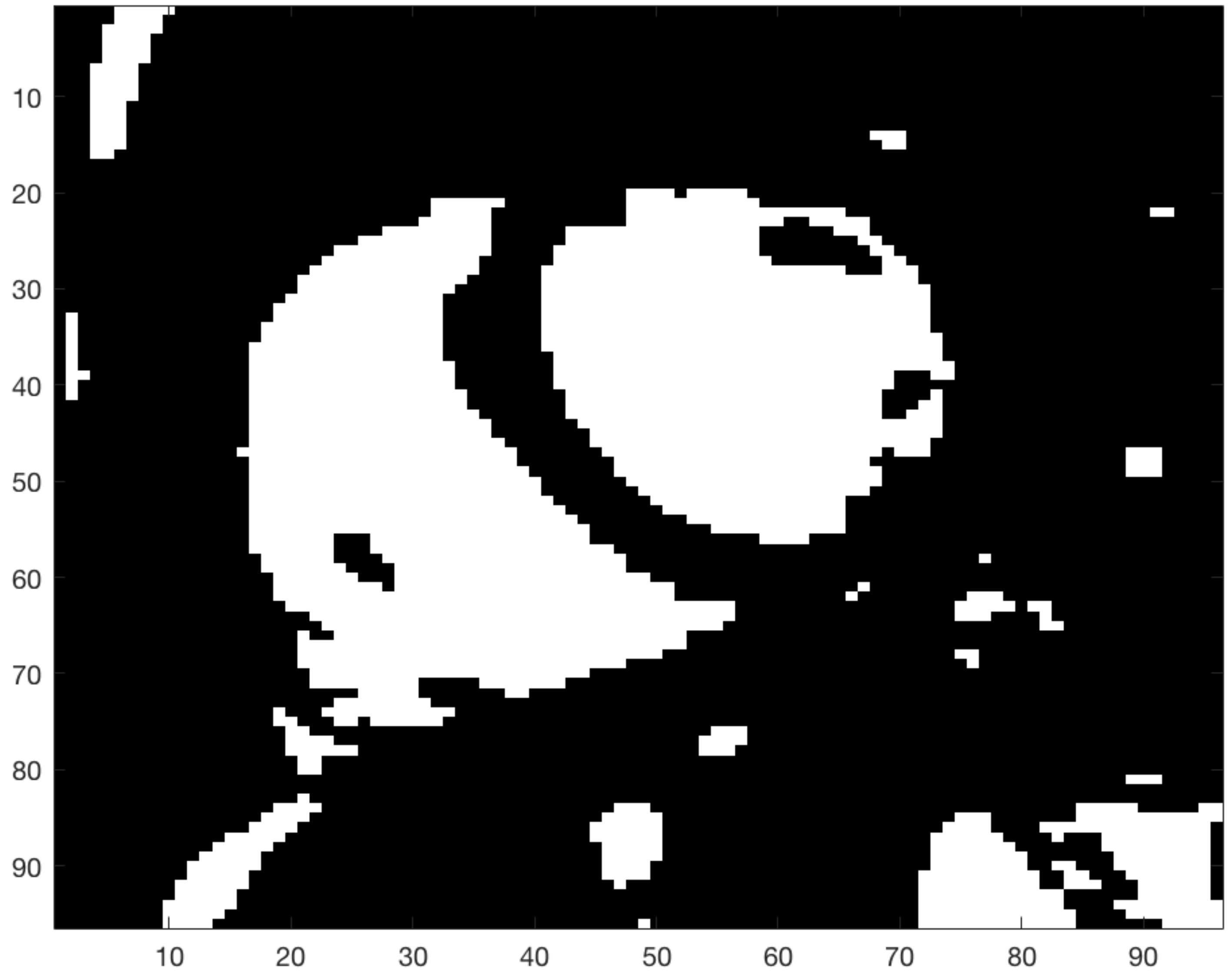
Discretize pixel brightness using 60 bins. Estimate probabilites

P(heart | x)



Discretize pixel brightness using 60 bins. Estimate probabilites

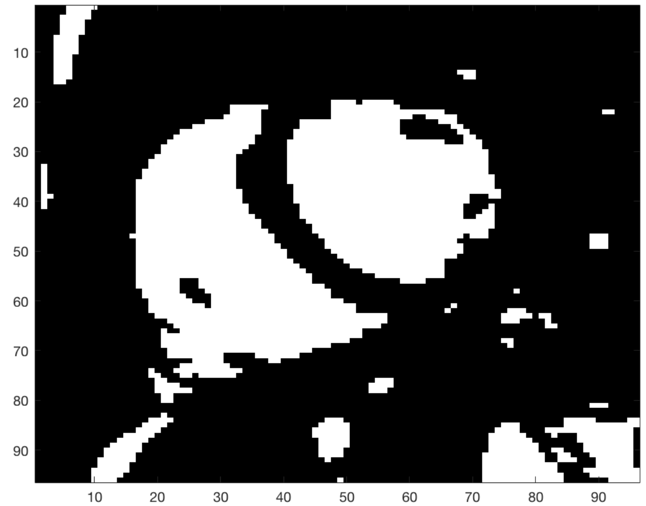
$P(\text{heart} \mid x) > 0.5$

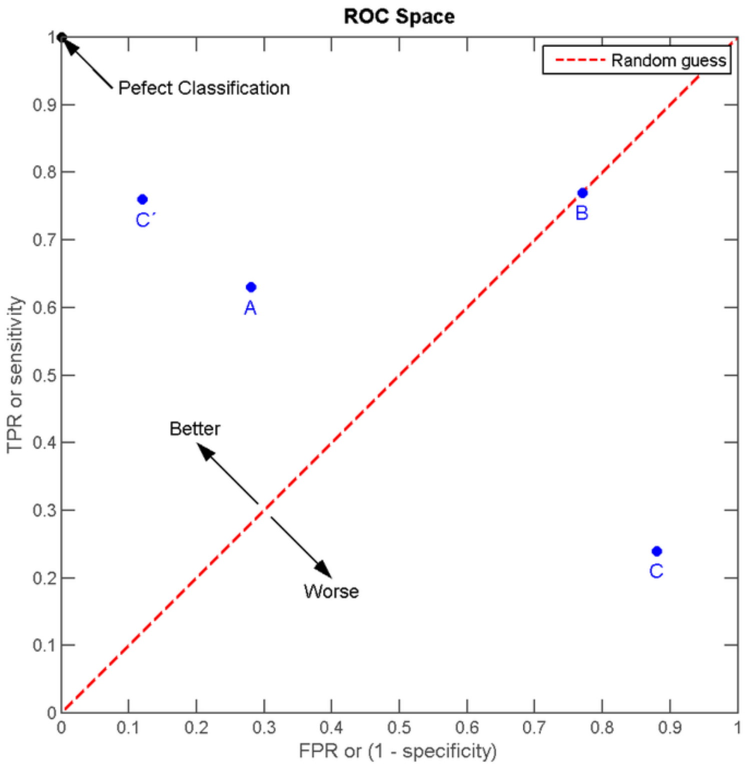


False Positives, False Negatives ROC - Curve

- For two class problems - Negatives and Positives
- Negatives that are classified as negatives – True Negatives (TN)
- Positives that are classified as positives – True Positives (TP)
- Negatives that are classified as positives – False Positives (FP)
- Positives that are classified as negatives – False Negatives (FN)

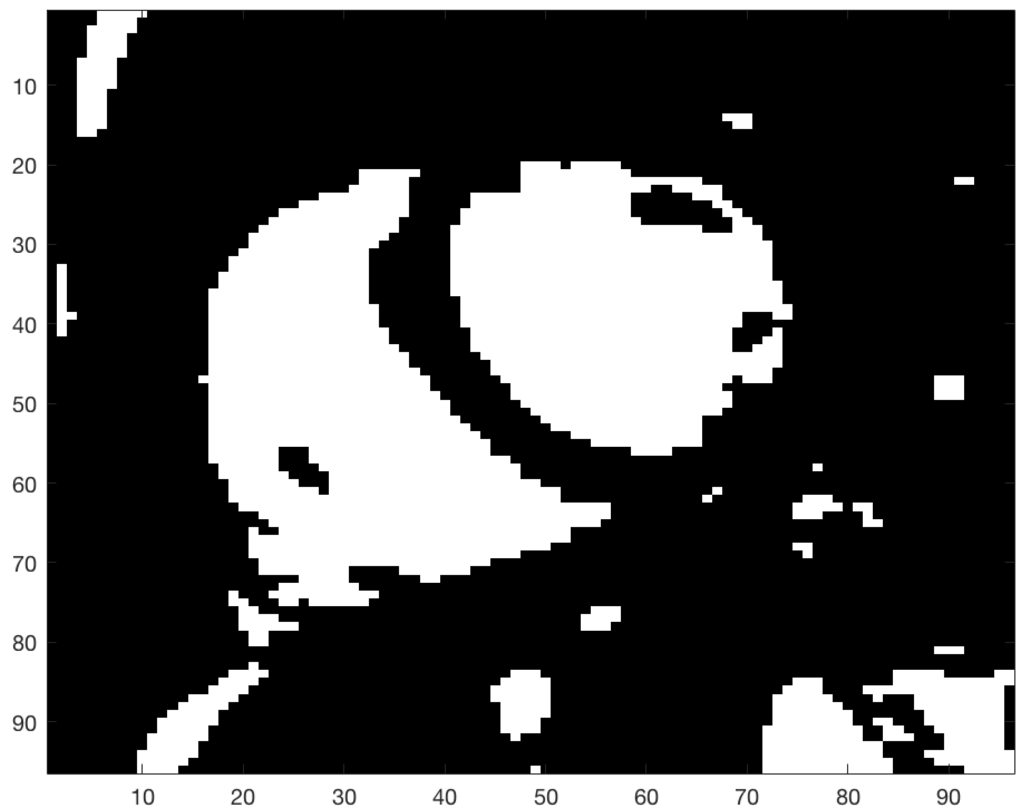
- False Positive Rate FPR = FP/(FP+TN) -> x-axis
- True Positive Rate TPR = TP/(TP+FN) -> y-axis

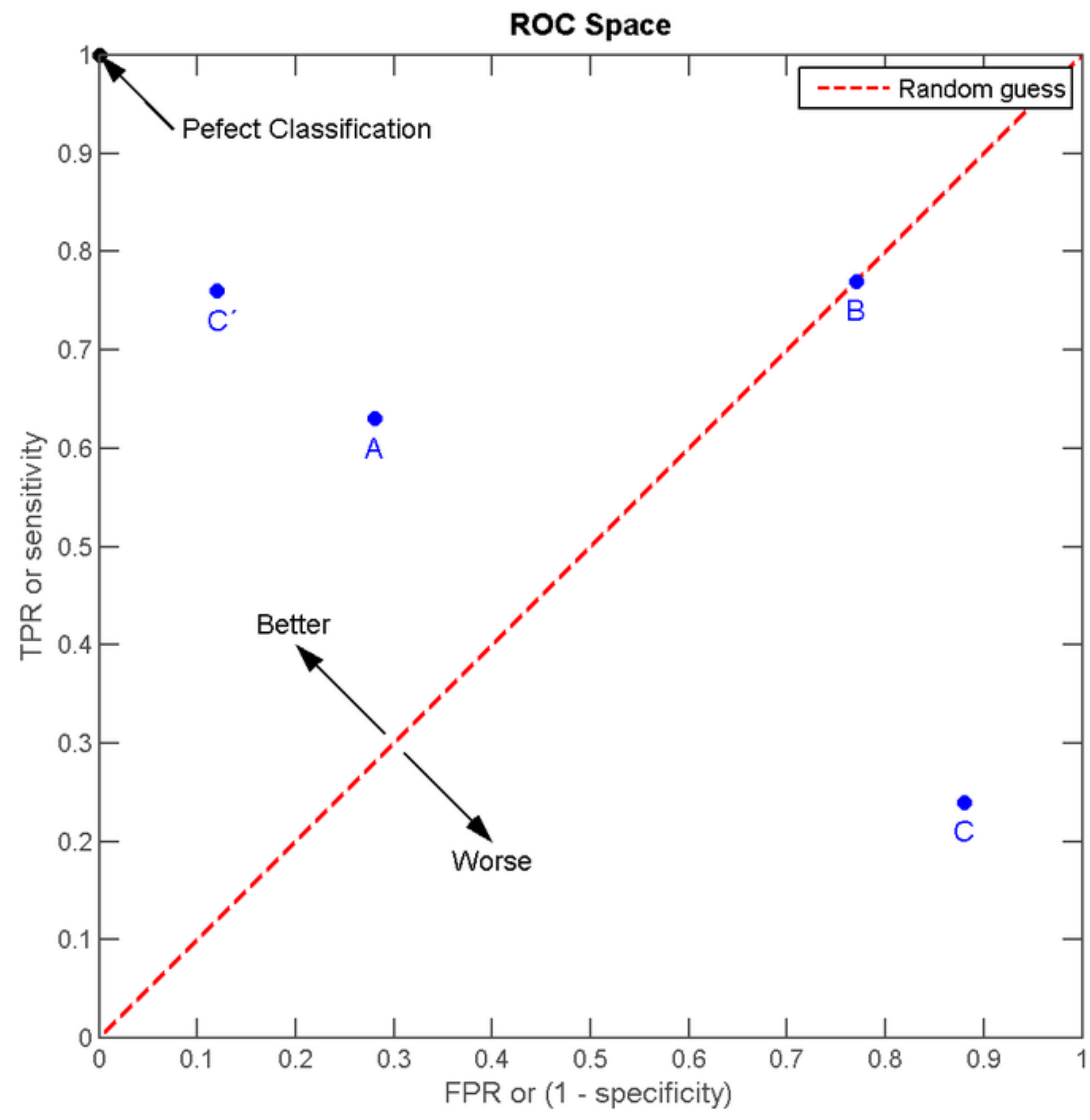




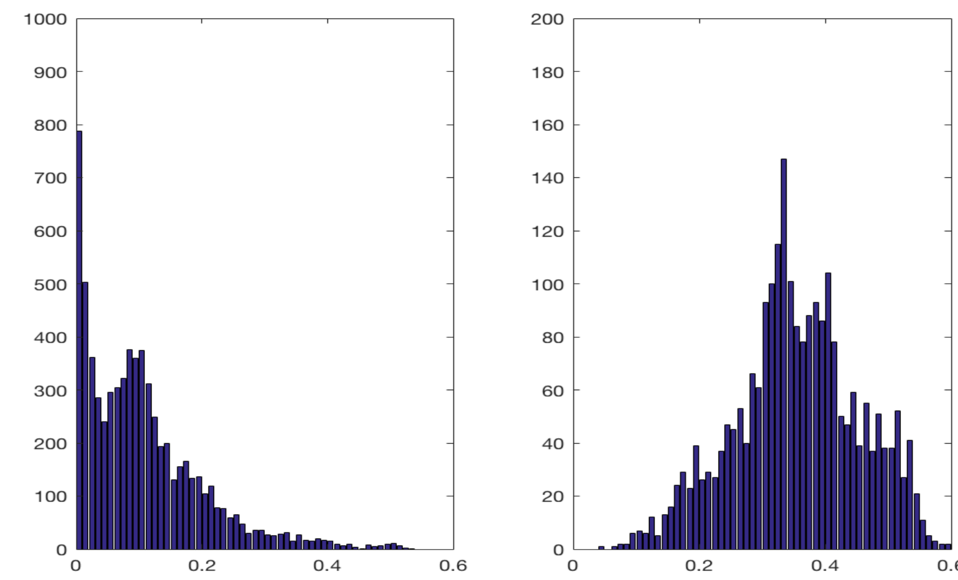
False Positives, False Negatives ROC - Curve

- FPR = FP/(FP+TN) -> x-axis
- TPR = TP/(TP+FN) -> y-axis





Bayes Theorem



- Bayes Theorem

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

- Interpret P as probabilites, e.g. If X and Y are discrete
- Interpret P as probability density functions, e.g. If X and/or Y are continuous stochastic variable,

$$P(Y=y|X=x) = \frac{f_X(x|Y=y)P(Y=y)}{f_X(x)}$$

$$f_Y(y|X=x) = \frac{f_X(x|Y=y)f_Y(y)}{f_X(x)}$$

From 1 channel to 3
Colour images

Binning in higher dimensions
The curse of dimensionality

- 10 bins in one dimension -> 10 bins
- 10 bins in two dimensions -> 100 bins
- 10 bins in three dimensions -> 1000 bins
- 10 bins in 30 dimensions -> 1 000 000 000 000 000 000 000 000 000 000 bins
- 10 bins in 128 dimensions -> 10^(128) bins

Richard Bellman

- ”*Many algorithms that work fine in low dimensions become intractable when the input is high-dimensional*.”, Bellman, 1961.

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Clustering – adaptive binning

- Colour images. Pixels are RGB with 8 bits each. 2^24 = 16777216 types of pixels.
- Too many.
- Try to bin in a more clever way?
- Clustering

Clustering: group together similar points and represent them with a single token

Key Challenges:

1) What makes two points/images/patches similar?

2) How do we compute an overall grouping from pairwise similarities?

Slide: Derek Hoiem

Why do we cluster?

- **Summarizing data**
  - Look at large amounts of data
  - Patch-based compression or denoising
  - Represent a large continuous vector with the cluster number
- **Counting**
  - Histograms of texture, color, SIFT vectors
- **Segmentation**
  - Separate the image into different regions
- **Prediction**
  - Images in the same cluster may have the same labels

Slide: Derek Hoiem

How do we cluster?

- K-means
  - Iteratively re-assign points to the nearest cluster center
- Agglomerative clustering
  - Start with each point as its own cluster and iteratively merge the closest clusters
- Mean-shift clustering
  - Estimate modes of pdf
- Spectral clustering
  - Split the nodes in a graph based on assigned links with similarity weights

K-means algorithm

1. Randomly select K centers

2. Assign each point to nearest center

3. Compute new center (mean) for each cluster

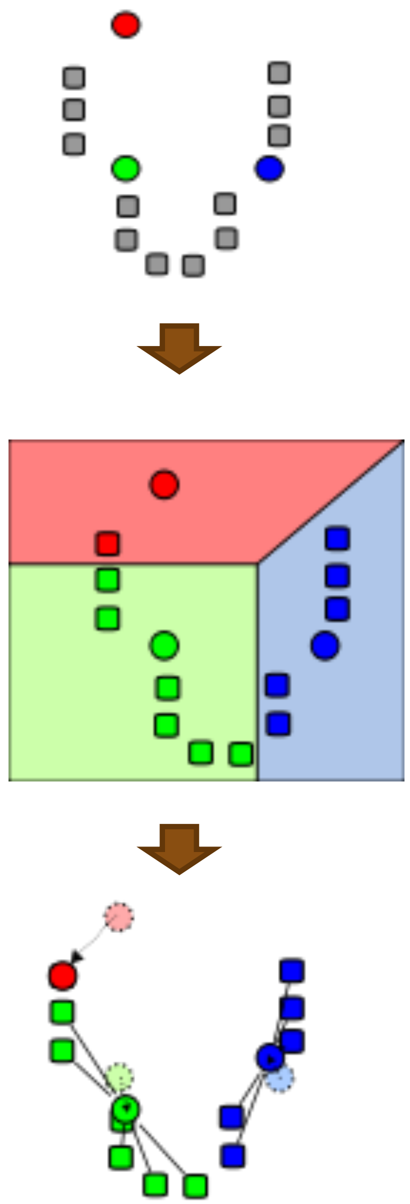


Illustration: http://en.wikipedia.org/wiki/K-means_clustering

K-means algorithm

1. Randomly select K centers

2. Assign each point to nearest center

3. Compute new center (mean) for each cluster

Back to 2

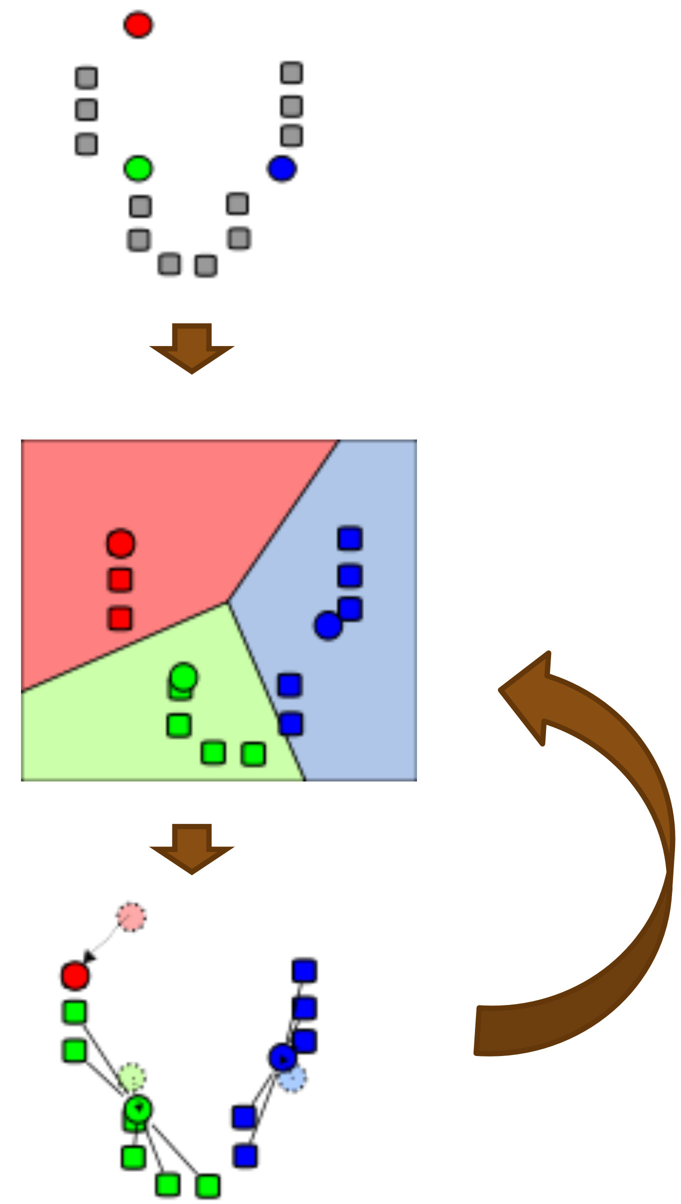


Illustration: http://en.wikipedia.org/wiki/K-means_clustering

K-means

1. Initialize cluster centers: $\mathbf{c}^0$; t=0

2. Assign each point to the closest center

$$\boldsymbol{\delta}^t = \underset{\boldsymbol{\delta}}{\operatorname{argmin}} \tfrac{1}{N} \sum_j^N \sum_i^K \delta_{ij} \left(\mathbf{c}_i^{t-1} - \mathbf{x}_j\right)^2$$

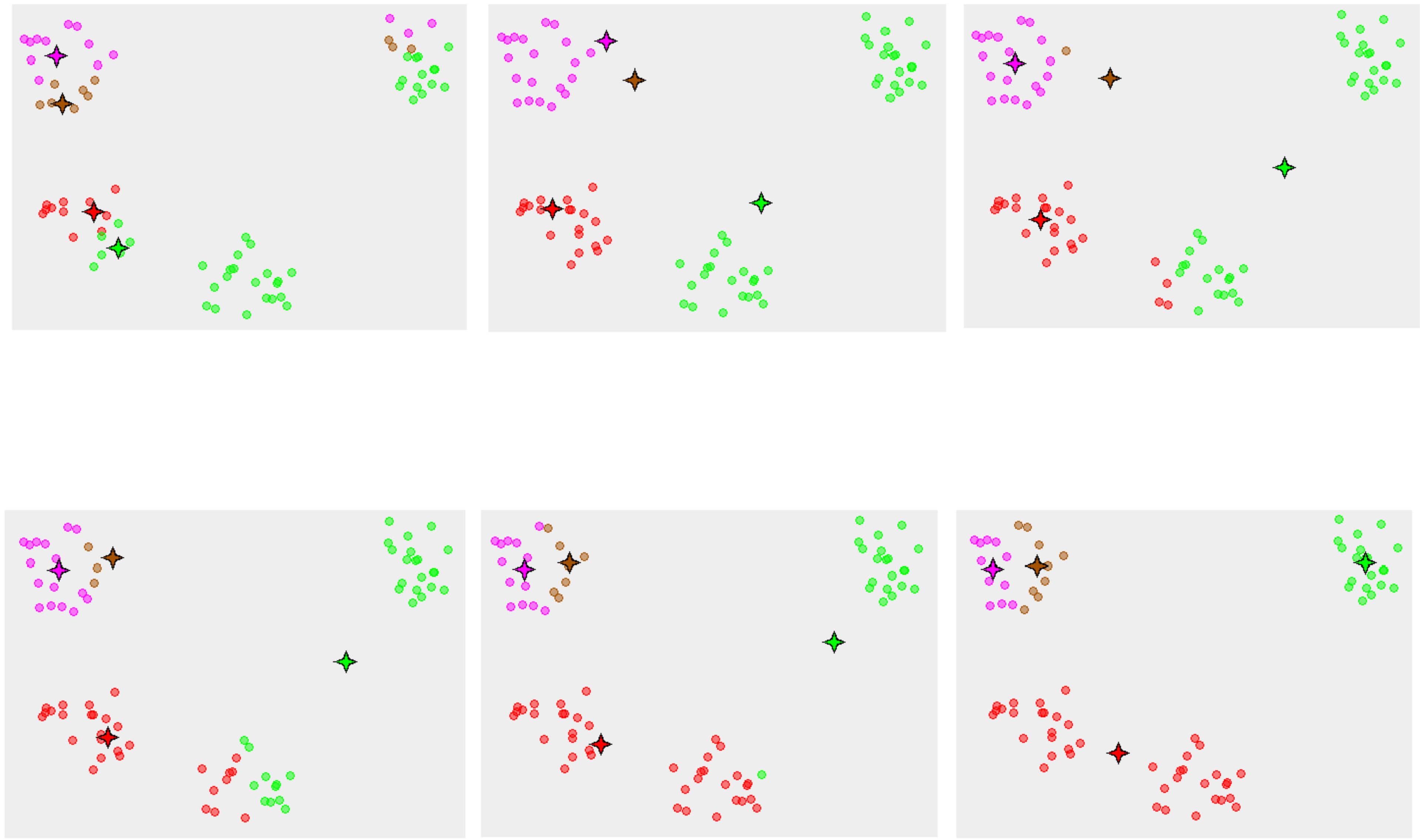
3. Update cluster centers as the mean of the points

$$\mathbf{c}^t = \underset{\mathbf{c}}{\operatorname{argmin}} \tfrac{1}{N} \sum_j^N \sum_i^K \delta_{ij}^t \left(\mathbf{c}_i - \mathbf{x}_j\right)^2$$

4. Repeat 2-3 until no points are re-assigned (t=t+1)

Slide: Derek Hoiem

K-means converges to a local minimum



K-means: design choices

- Initialization
  - Randomly select K points as initial cluster center
  - Or greedily choose K points to minimize residual

- Distance measures
  - Traditionally Euclidean, could be others

- Optimization
  - Will converge to a *local minimum*
  - May want to perform multiple restarts

How to evaluate clusters?

- Generative
  - How well are points reconstructed from the clusters?
- Discriminative
  - How well do the clusters correspond to labels?
  - Note: unsupervised clustering does not aim to be discriminative

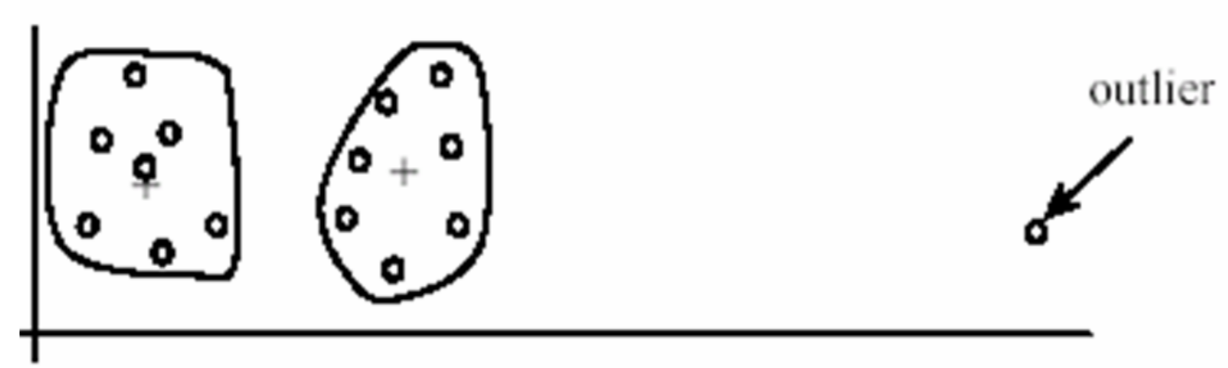
Slide: Derek Hoiem

How to choose the number of clusters?

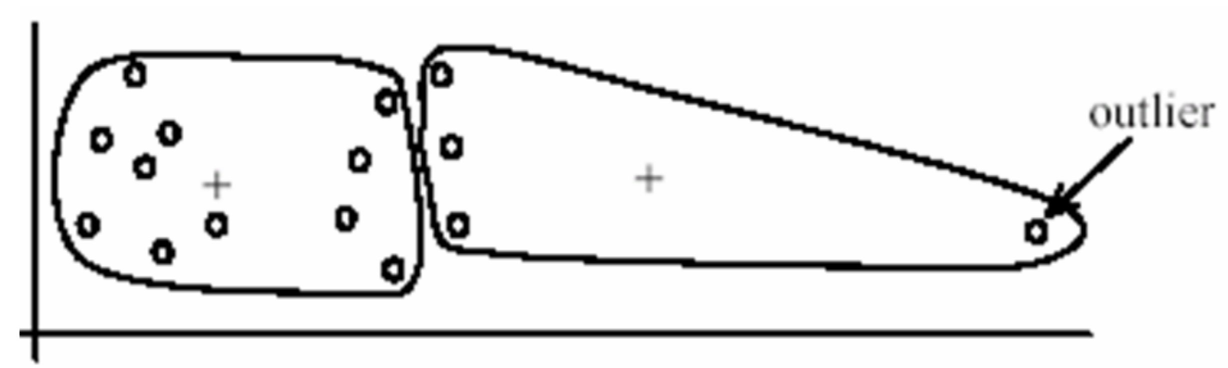
- Validation set
  - Try different numbers of clusters and look at performance
    - When building dictionaries (discussed later), more clusters typically work better

K-Means pros and cons

- Pros
  - Finds cluster centers that minimize conditional variance (good representation of data)
  - Simple and fast*
  - Easy to implement
- Cons
  - Need to choose K
  - Sensitive to outliers
  - Prone to local minima
  - All clusters have the same parameters (e.g., distance measure is non-adaptive)
  - *Can be slow: each iteration is O(KNd) for N d-dimensional points
- Usage
  - Rarely used for pixel segmentation

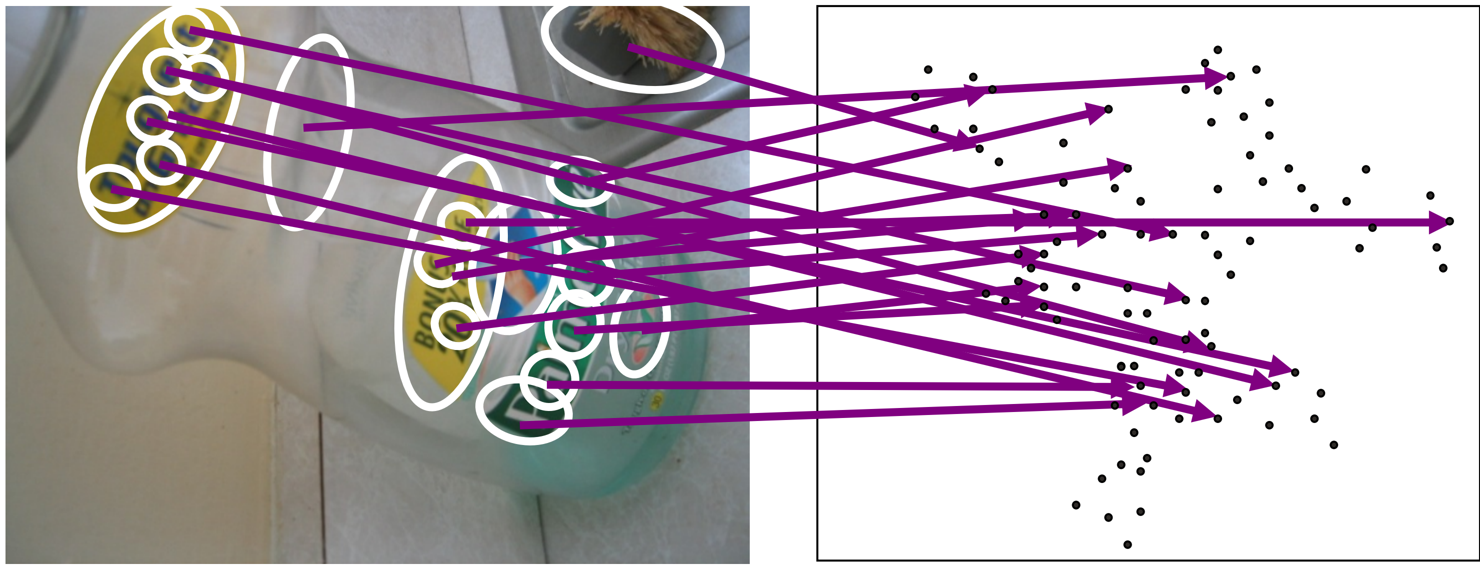


(B): Ideal clusters

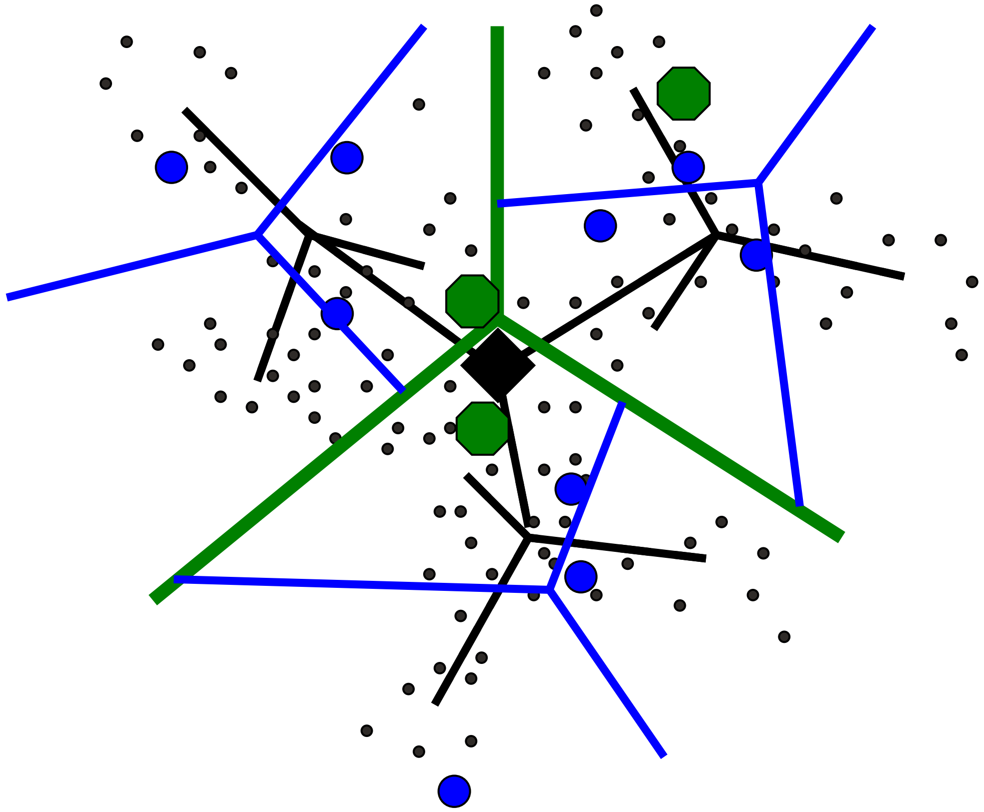


Building Visual Dictionaries

1. Sample patches from a database
   - E.g., 128 dimensional SIFT vectors

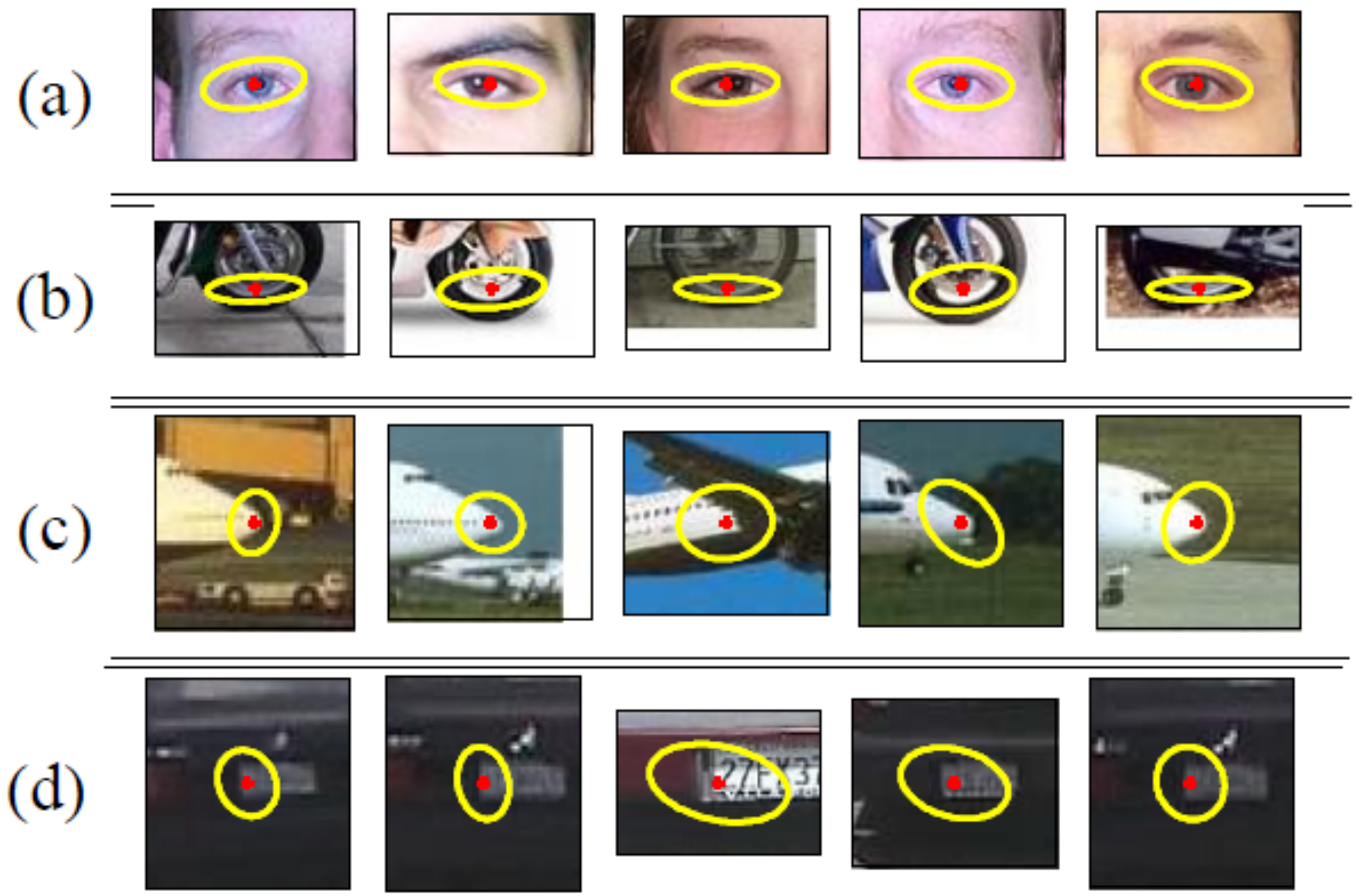


2. Cluster the patches
   - Cluster centers are the dictionary



3. Assign a codeword (number) to each new patch, according to the nearest cluster

Examples of learned codewords



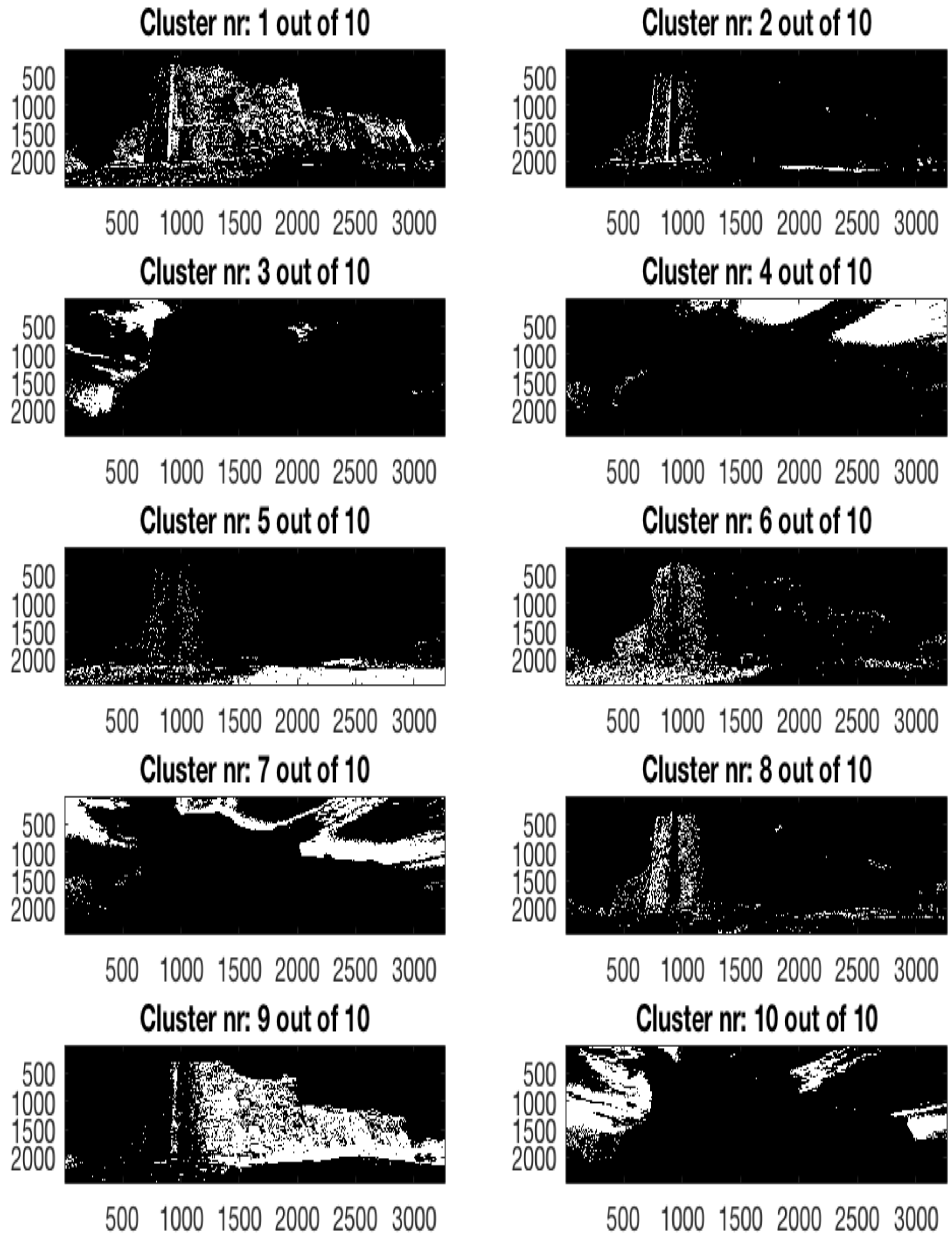
Most likely codewords for 4 learned "topics"
EM with multinomial (problem 3) to get topics

http://www.robots.ox.ac.uk/~vgg/publications/papers/sivic05b.pdf Sivic et al. ICCV 2005

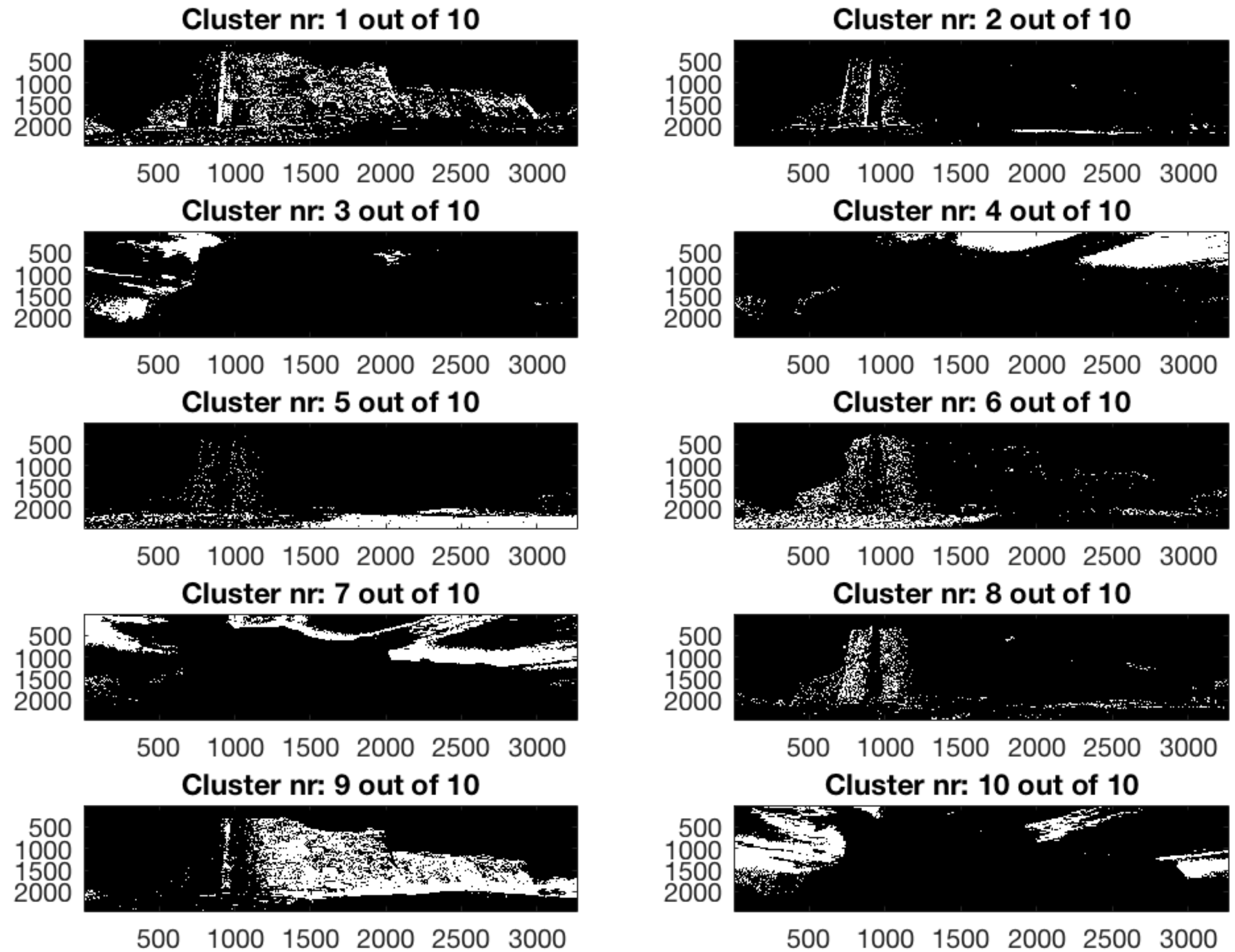
Example: Colour pixel classification

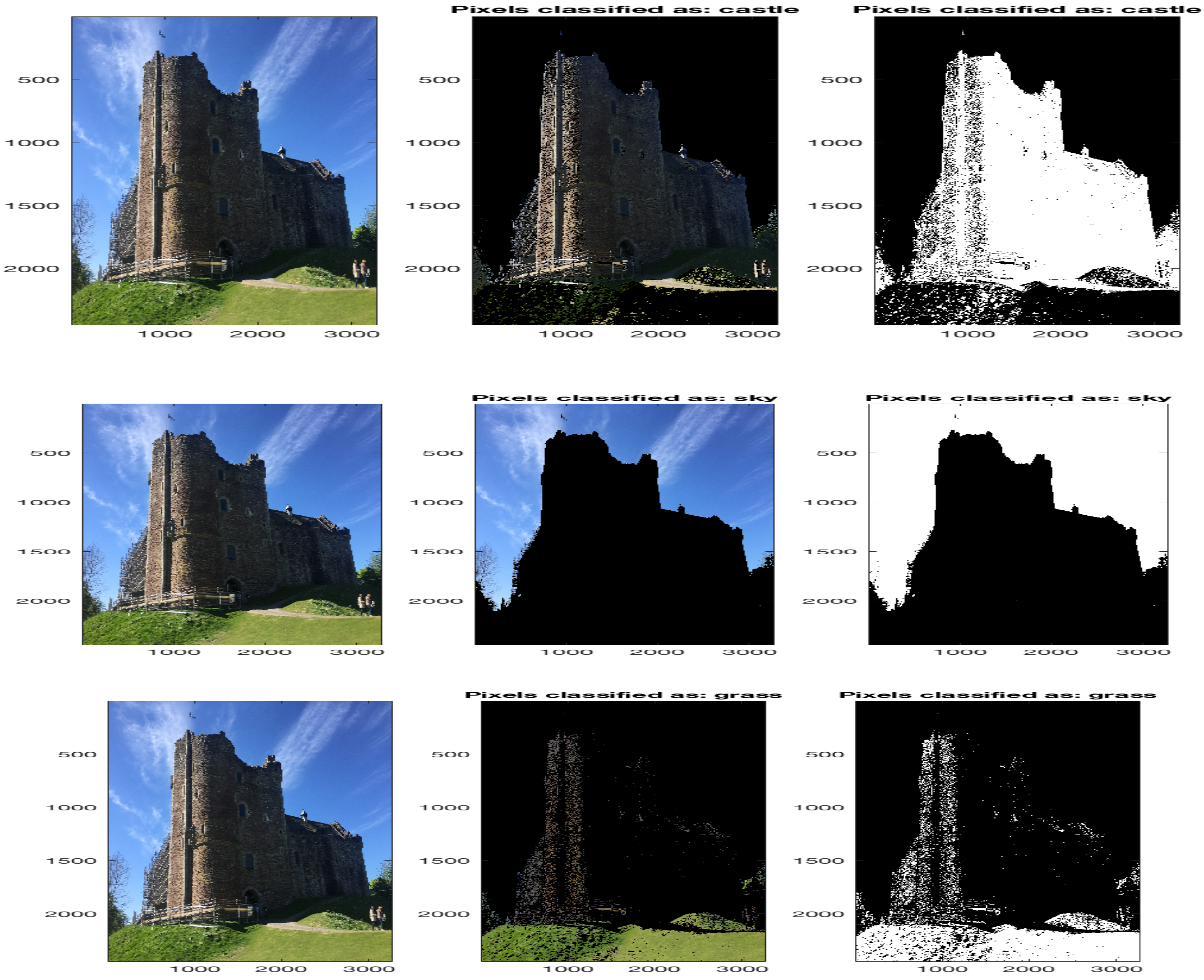
- Use clustering to assign codewords (bin nr) x 1-10 for each pixel
- Estimate measurement probabilites p(x|y) for each class y (1-grass, 2-castle, 3-sky) and each bin x.
- Use Bayes theorem to calculate p(y|x) for each pixel,
- Classify according to maximum a posteriori probability

Example: Colour pixel classification



Example: Colour pixel classification

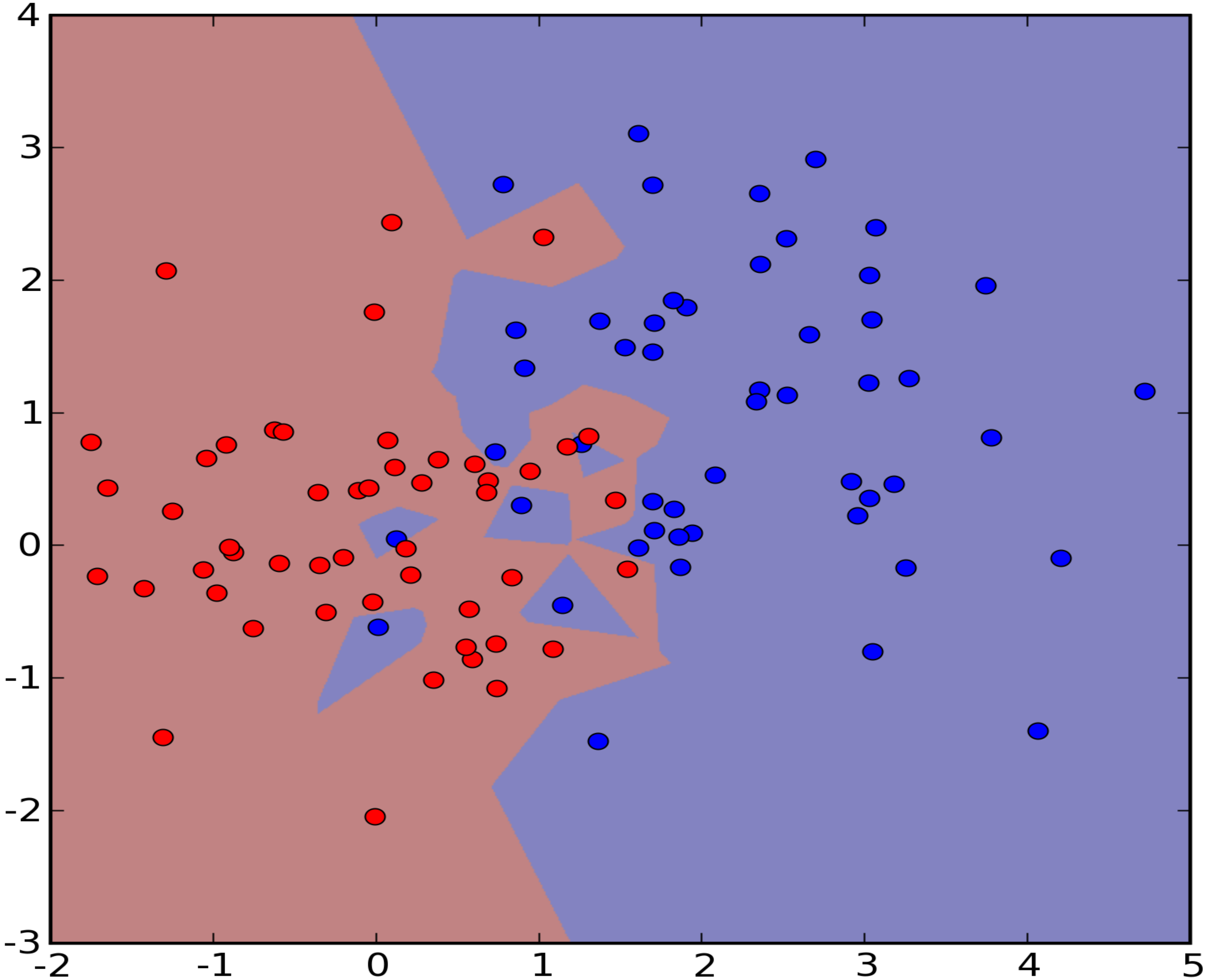



Pixels classified as: castle
Pixels classified as: castle
Pixels classified as: sky
Pixels classified as: sky
Pixels classified as: grass
Pixels classified as: grass
500
1000
1500
2000
1000
2000
3000

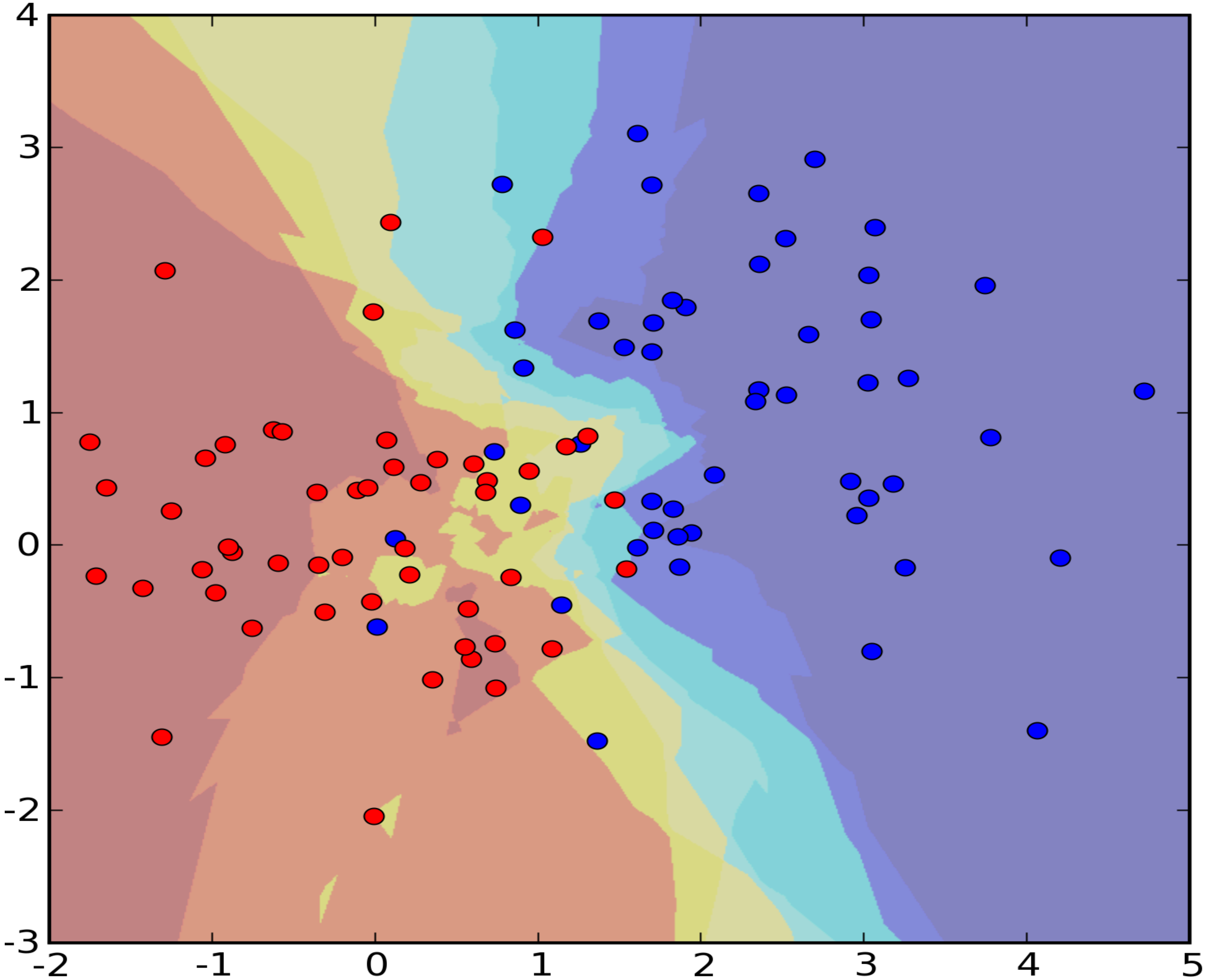
Nearest Neighbour Classification NN and K-NN

- Classify using training data (x_i, y_i)
- NN: Use the label of the nearest neighbour
- KNN: Use the label of the majority of the k nearest neigbhours
- Regression: Use the average of the value of the k nearest neighbours
- Easy to implement and understand
- Can use arbitrary distance functions between images
- Converges to the optimum
- Slow when using lots of data, need to store all training data, not smooth regression

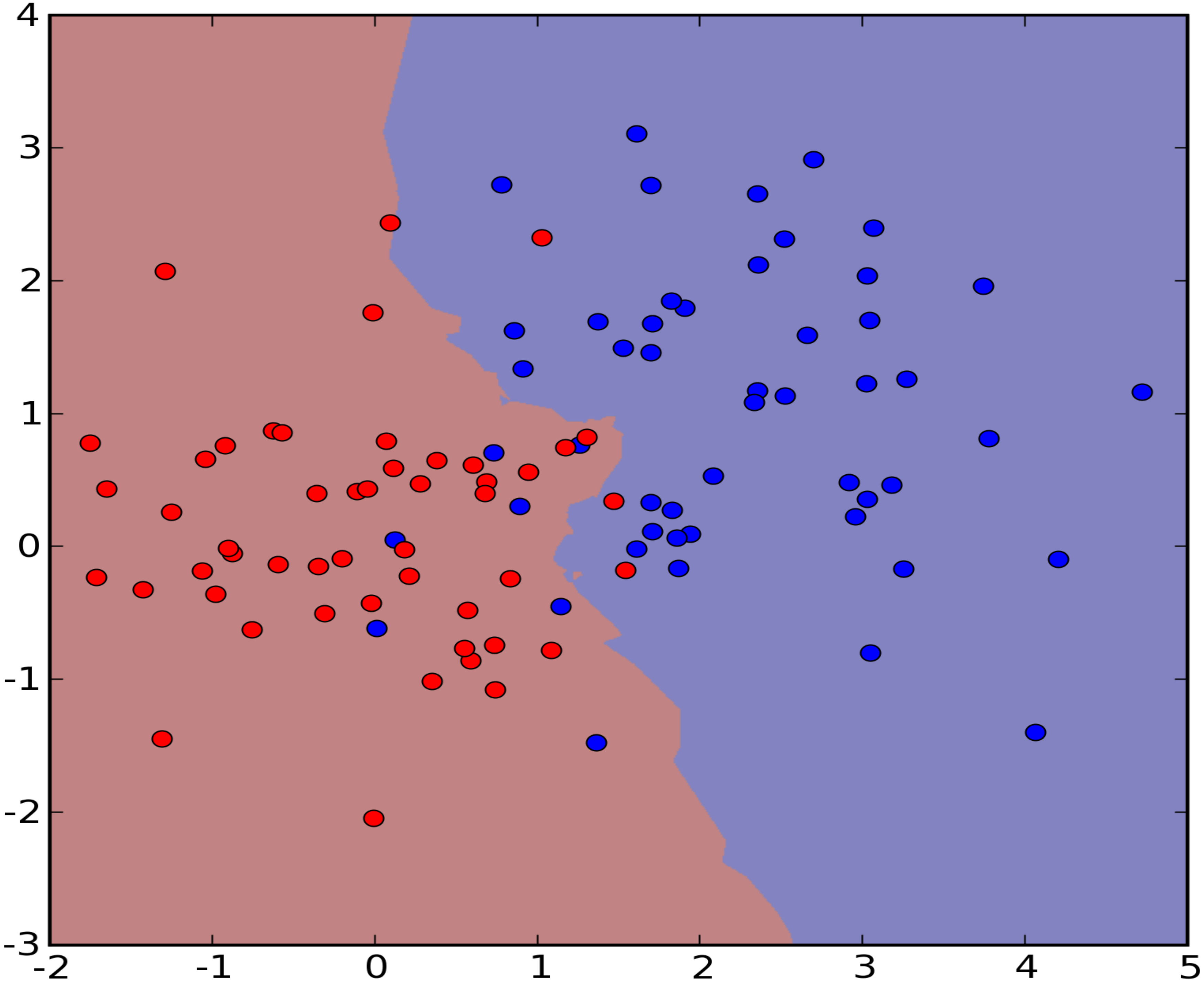
Nearest Neighbour Classification (discussion)



7 Nearest Neighbour Classification



7 Nearest Neighbour Classification



Nearest Neighbour Classification NN and K-NN

- Training is easy, just store the training data $T = \{(x_1,y_1), \ldots, (x_N,y_N)\}$
- Works in any dimension
- Works for regression also: Use the average of the value of the k nearest neighbours
- Easy to implement and understand
- Can use arbitrary distance functions between images
- Converges to the optimum

- Slow when using lots of data,
- Need to store all training data
- Not smooth regression

Parametric density estimation Plug-in Classifier

- **Parametric density estimation**
- (compare with Nonparametric density estimation)
  - Parametric – fixed nr of parameters
  - Nonparametric – nr of parameters grow with training data
- Plug-in classifier, i.e. plug-in the estimated densities in Bayes rule
- Classification

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Parametric density estimation
Plug-in Classifier

- Bayes Theorem

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

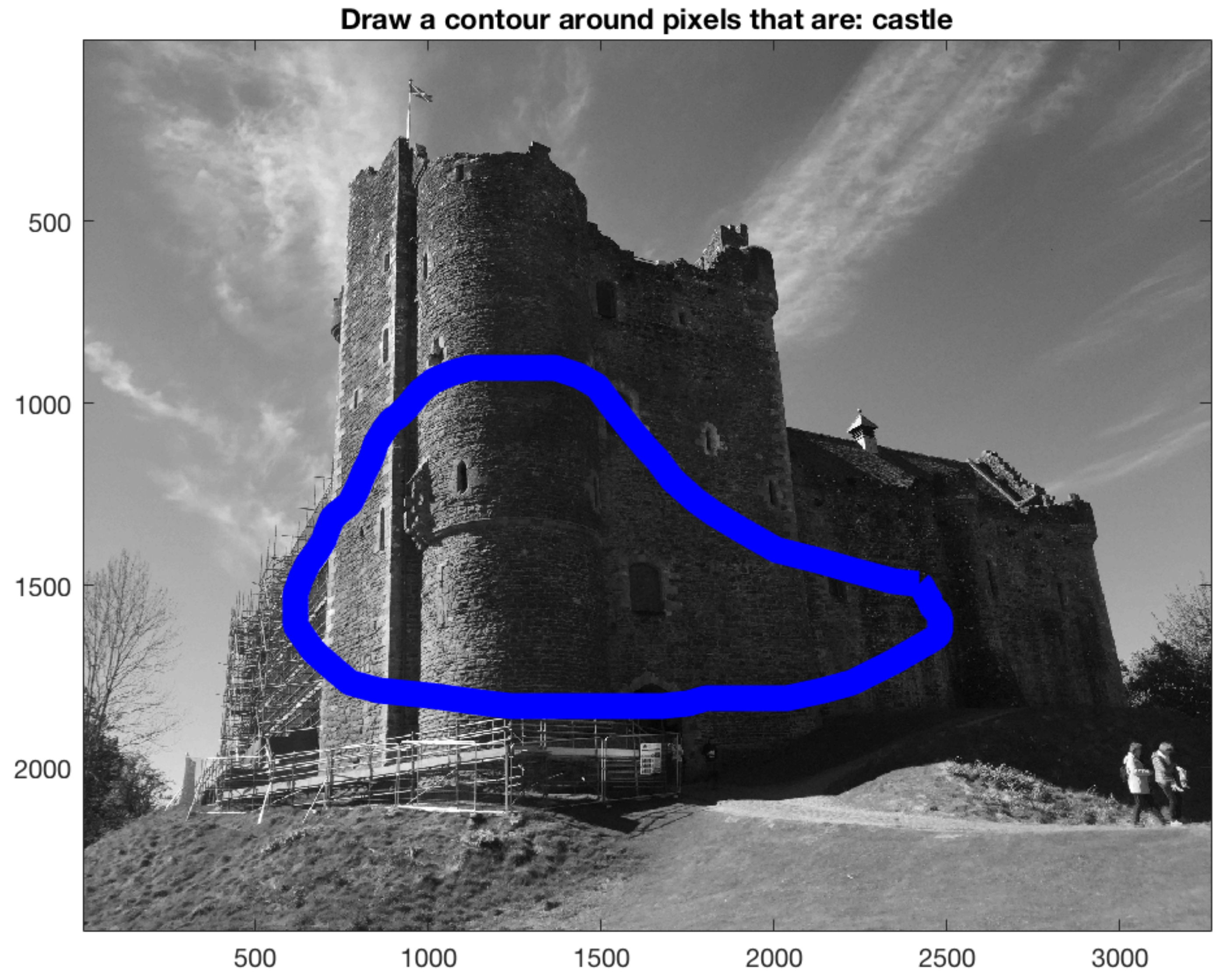
- Interpret P as probabilites, e.g. If Y – discrete
- Interpret P as probability density functions, e.g. If X and/or Y are continuous stochastic variable,

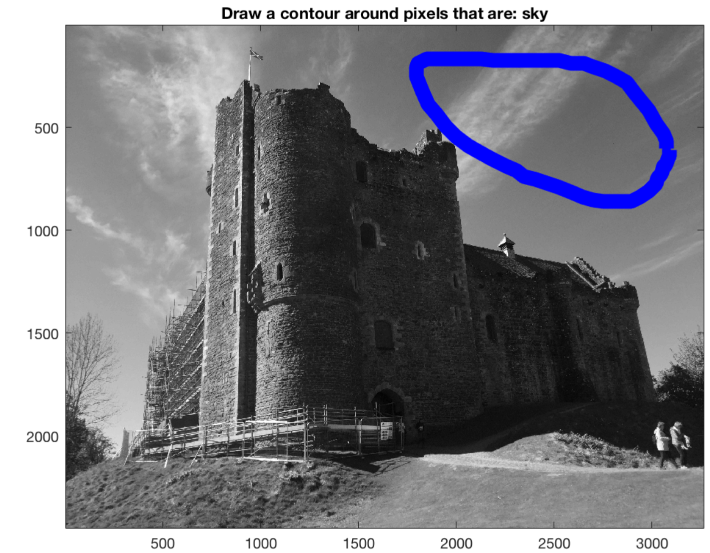
$$P(Y = y|X = x) = \frac{f_X(x|Y = y)P(Y = y)}{f_X(x)}$$

$$f_Y(y|X = x) = \frac{f_X(x|Y = y)f_Y(y)}{f_X(x)}$$

Parametric density estimation

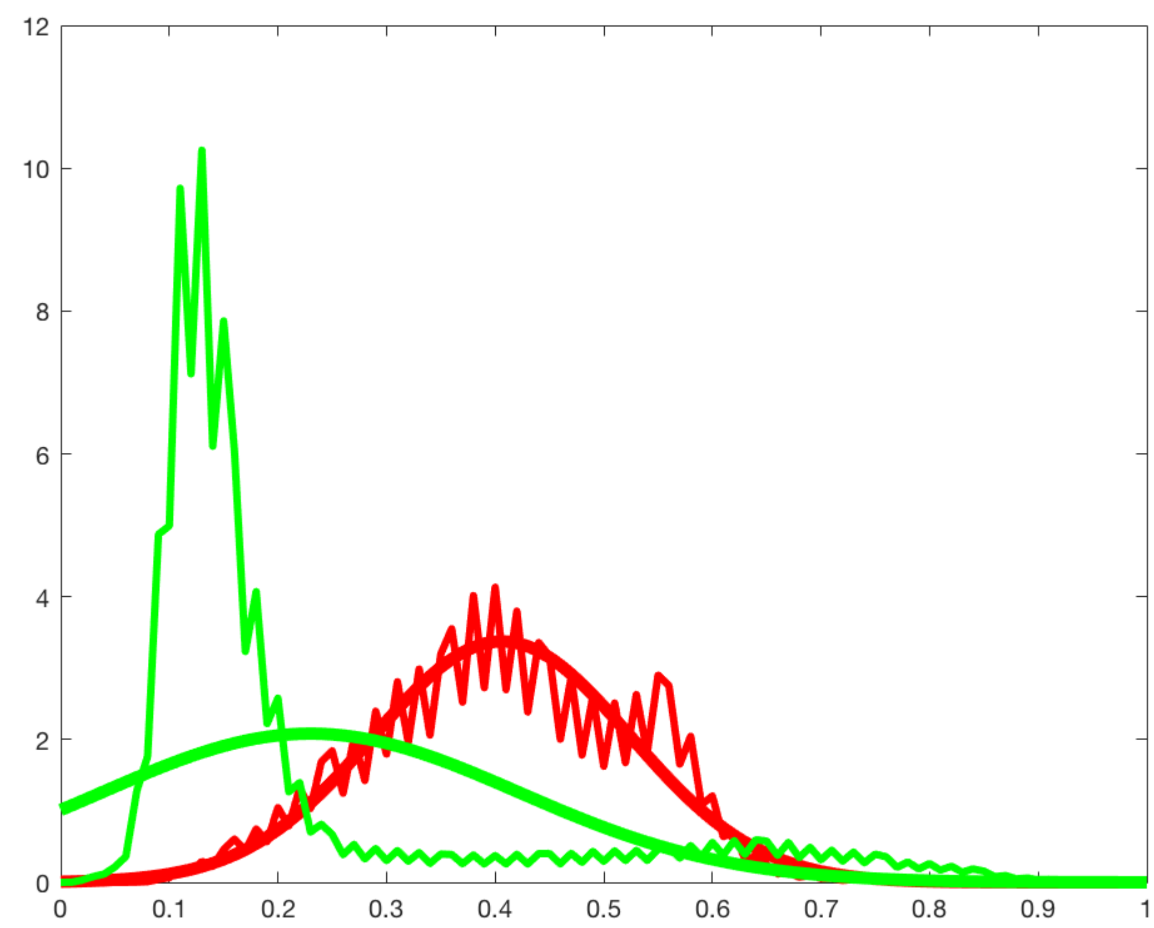
Example: Pixel classification. Gray-scale image, easier to plot 1D distributions.
Classify 'Castle' vs ('sky' and 'grass')

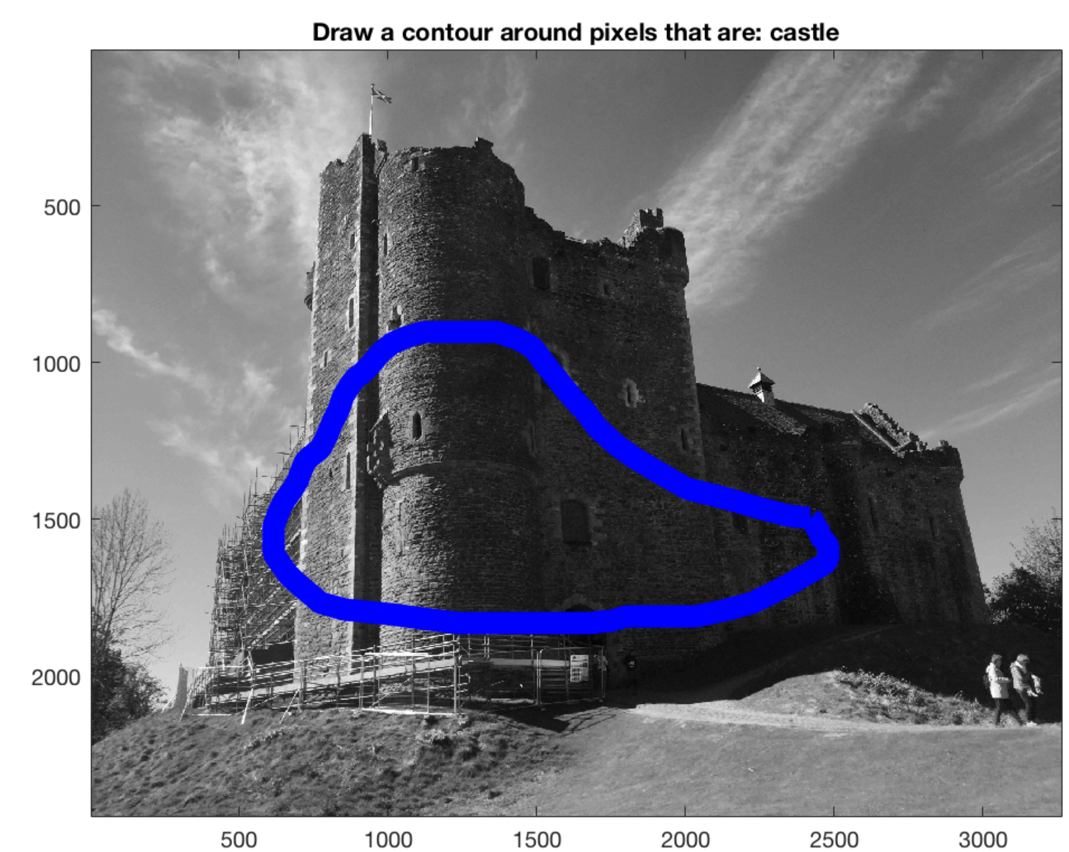




Parametric density estimation

'Castle' vs ('sky' and 'grass')
Since it is 1D. Easy to to both binning and parametric density estimation
Assume normal distribution.
Two parameters, mean m and standard deviation sigma.
Estimate using maximum likelihood method.

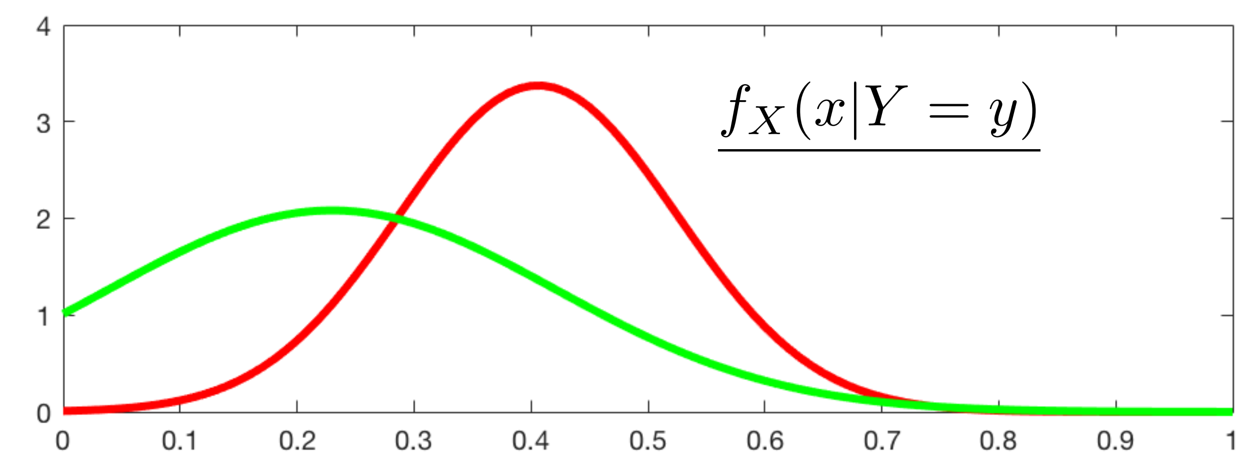




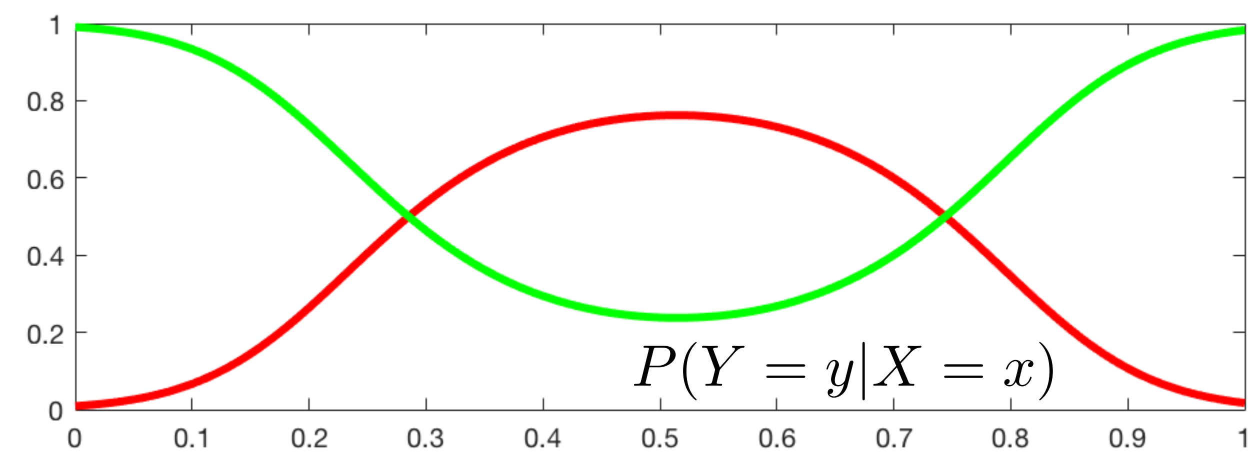
Parametric density estimation

1. Parametric density estimation – estimate m and sigma, (estimate or guess prior)
2. Plug-in classifier, i.e. plug-in the estimated densities (and priors) in Bayes rule

$$P(Y=y|X=x) = \frac{f_X(x|Y=y)P(Y=y)}{f_X(x)}$$

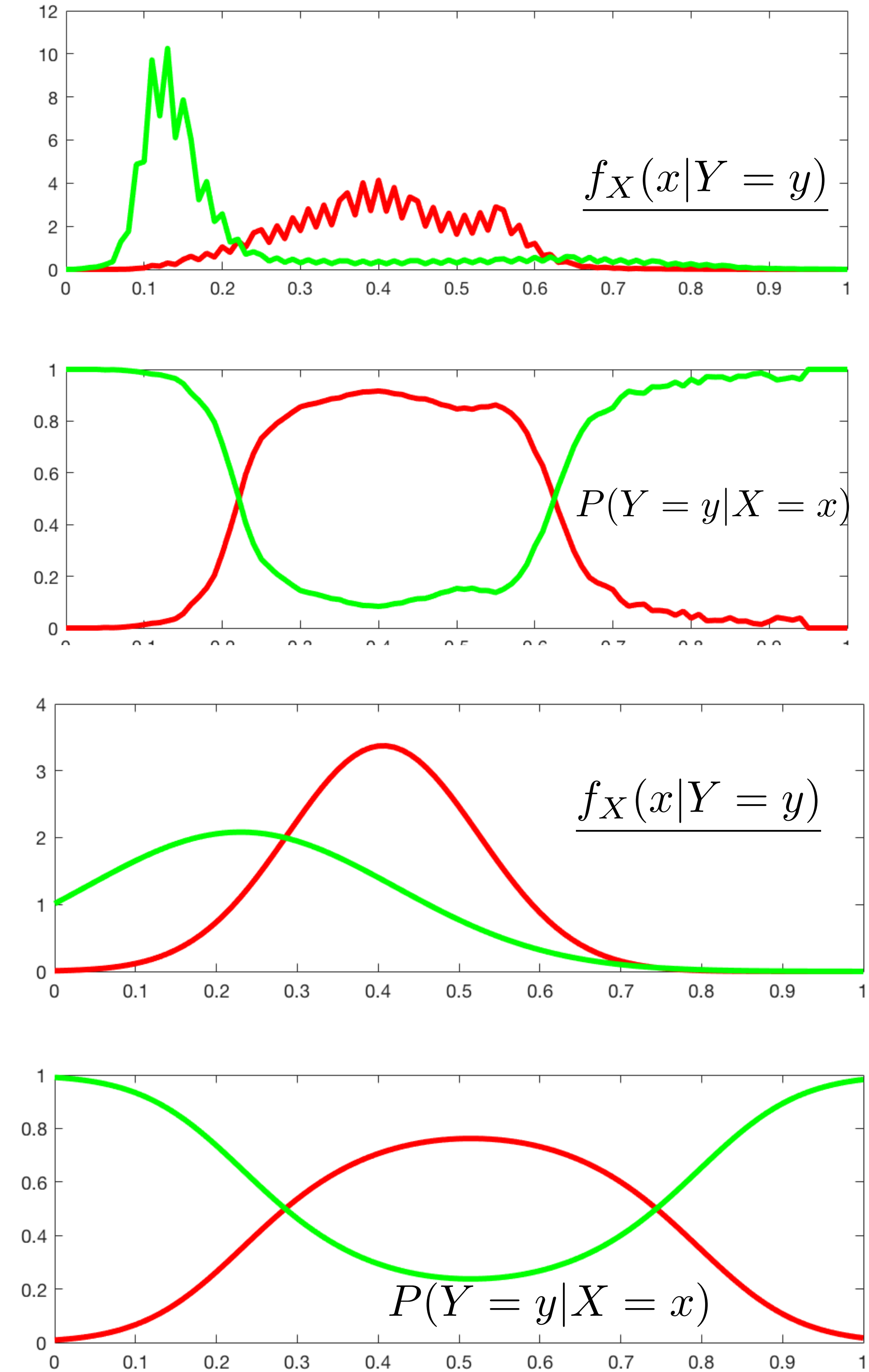


3. Classification



Compare binning Vs parametric

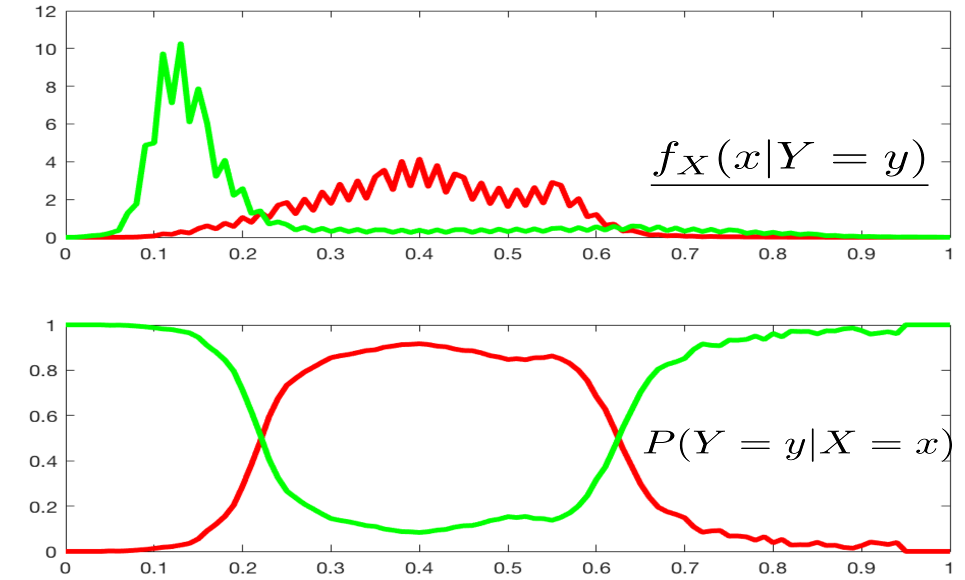
1. Binning
   1. Works only in 1 or a few dimensions
   2. Can be used in higher dimensions with adaptive binning (clustering, k-means)
   3. Discontinuous
2. Parametric density estimation
   1. Must guess density model
   2. Fewer parameters to estimate
   3. Fewer parameters to store
   4. Can be smooth



Parametric density estimation

- Many interesting and useful parametric densities to choose from
- Estimation of parameters can be
  - easy (simple formulas) or
  - hard
    - involve convex optimization (no local optima, guaranteed results
    - non-convex optimization (many local optima, no guarantee on finding the best optima.
- Fewer parameters to estimate
- Fewer parameters to store
- Can be smooth

Logistic regression



- Motivation
- In the end we are only interested in the posterior distribution

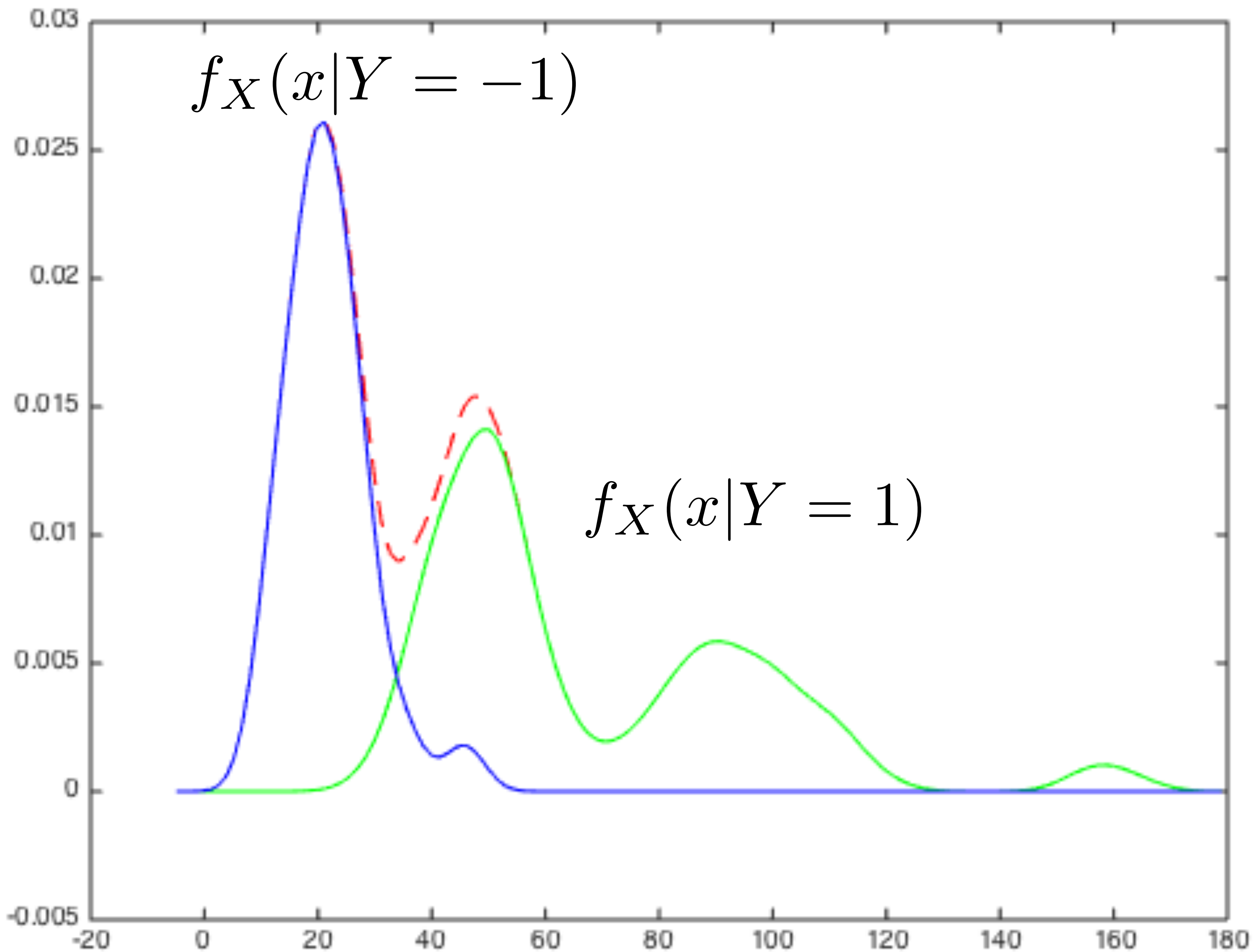
$$P(Y = y|X = x)$$

- Why not estimate this instead
- Skip the step of estimating the measurement densities

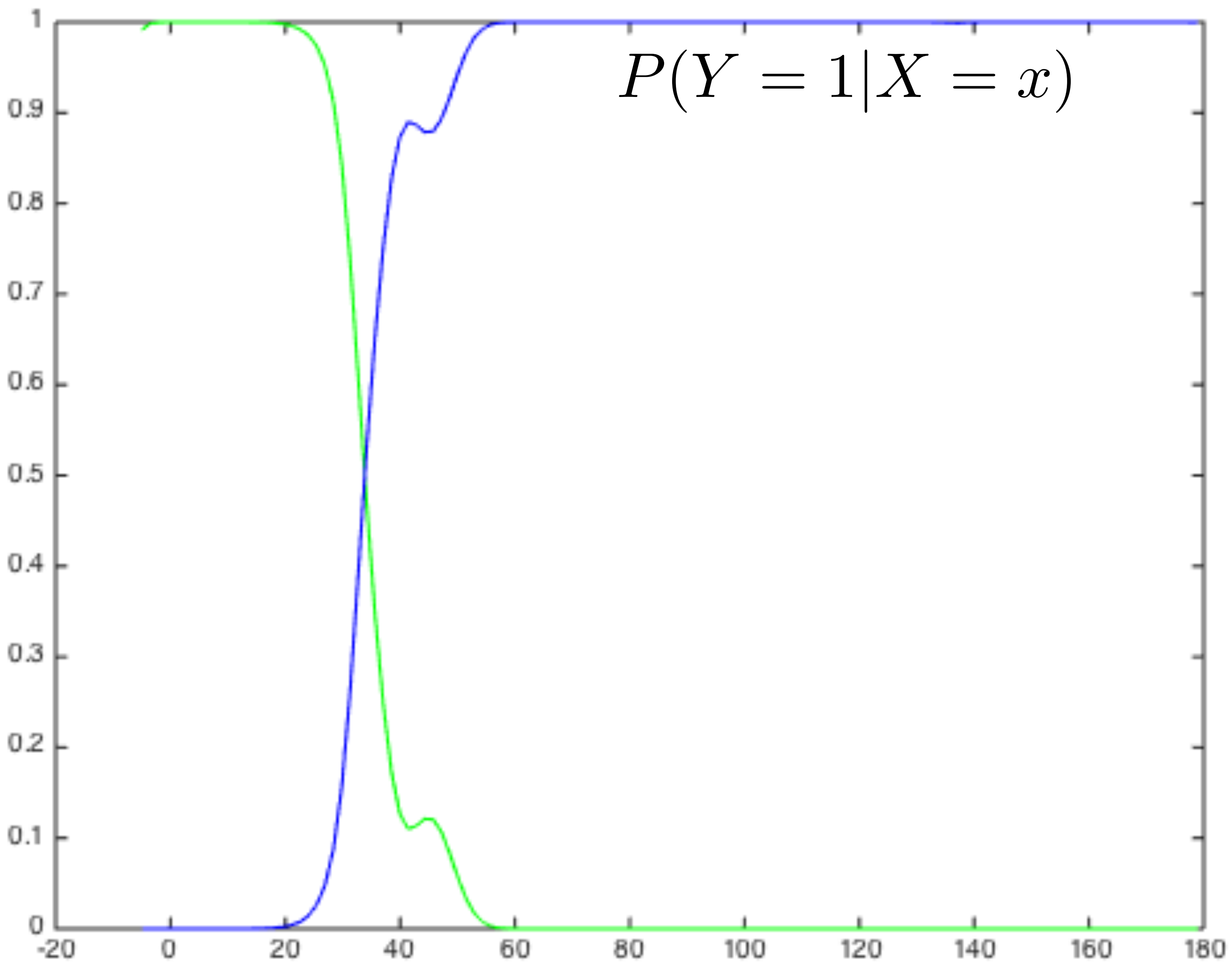
$$\underline{f_X(x|Y = y)}$$

- Details far away from the transition points are uninteresting (perhaps)
- Notice that the posterior looks like a smoothed step function

Example $P(Y=y|X=x) = \frac{f_X(x|Y=y)P(Y=y)}{f_X(x)}$



Example $P(Y=y|X=x) = \frac{f_X(x|Y=y)P(Y=y)}{f_X(x)}$



Logistic regression

- Discuss ideas and derivations on blackboard
- z = simple function of x, e.g. Linear z = $w^T x+b$
- Output y = smooth threshold of z, for example

$$s(z) = \frac{1}{1+e^{-z}}$$

- Notice that s(z) looks like a typical P(Y=1 | x) function

$$x \in R^d, w \in R^d, b \in R, f(x) = s(w^T x + b)$$

$$P(Y = 1|x) = \frac{1}{1+e^{-z}}$$

Derivation

- Estimate parameters

$$P(Y=1|x) = \frac{1}{1+e^{-z}}$$

$$P(Y=-1|x) = 1 - \frac{1}{1+e^{-z}} = \frac{e^{-z}}{1+e^{-z}} = \frac{1}{e^{z}+1}$$

- For both cases we have

$$P(Y=y|x) = \frac{1}{1+e^{-yz}}$$

- Calculate likelihood for training data

$$T = (x_1, y_1), \ldots, (x_n, y_n)$$

Estimate parameters

- Parameters $\theta = (w, b)$

$$P(Y = y|x) = \frac{1}{1 + e^{-yz}}$$

$$T = (x_1, y_1), \ldots, (x_n, y_n)$$

$$log(P) = log(\prod_i P(Y = y_i | x_i, \theta))$$

$$\text{max} \quad \sum_i \log(\frac{1}{1 + e^{y_i(w^T x_i + b)}})$$

$$\min_{\boldsymbol{w}} \quad \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + C\sum_{i=1}^{l} \log(1 + e^{-y_i \boldsymbol{w}^T \boldsymbol{x}_i}).$$ (dual problem)

Logistic regression

Standard logistic function

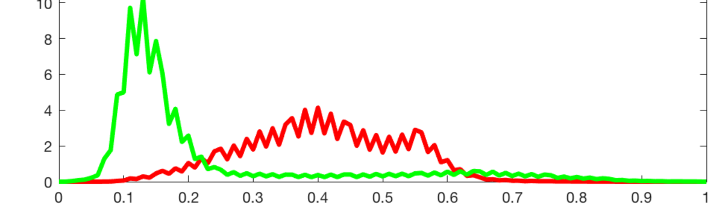
$$s(z) = \frac{1}{1 + e^{-x}}$$

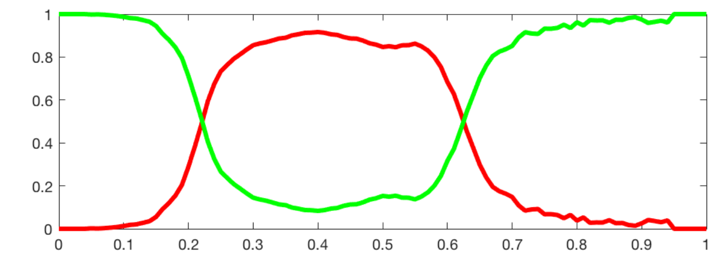
- Linear logistic regression
- Estimate the posterior $P(Y = y|X = x)$
- As linear function followed by standard logistic function

$$s(w^T x + b)$$

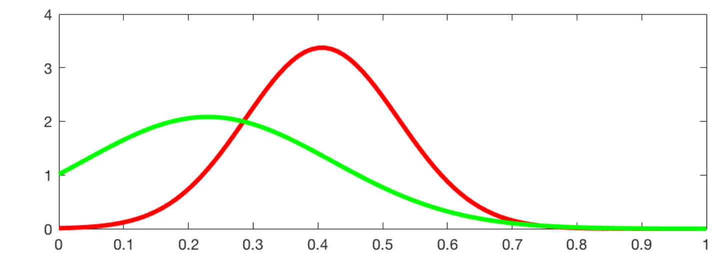
- Convex optimization problem

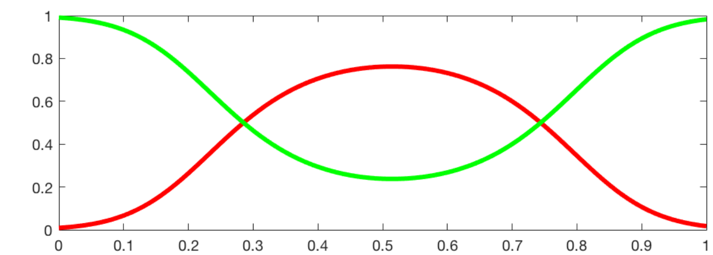
Logistic regression





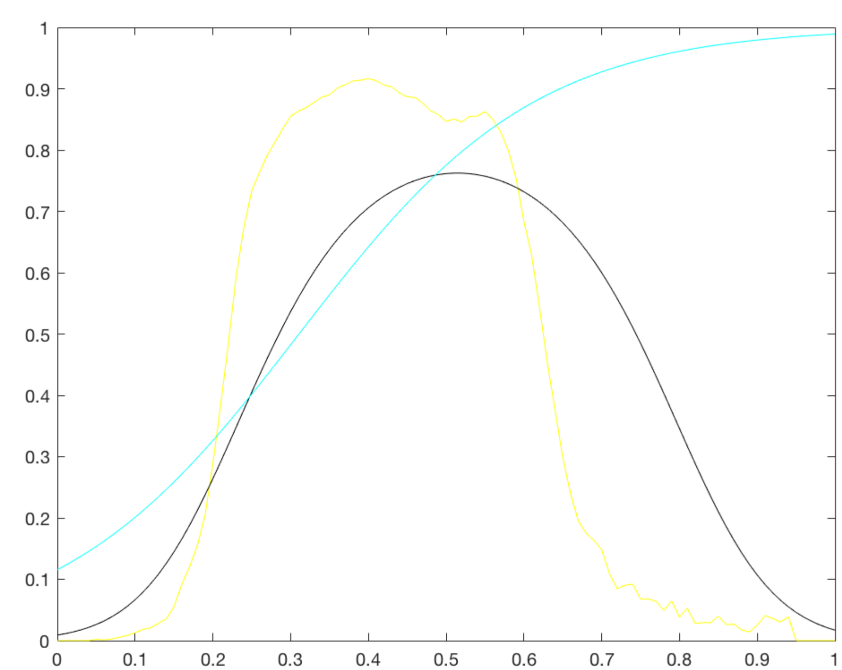
- Estimate w and b
- Posterior distribution $P(Y = y|X = x)$
- Simple model $s(w^T x + b)$





Standard logistic function

$$s(z) = \frac{1}{1 + e^{-x}}$$



Review

- Machine Learning
  - Classification
    - Bayes Rule
    - Estimating density functions
      - Counting
      - Binning
      - Adaptive binning (k-means)
      - Parametric density estimation
      - Plug in estimated densities
    - **Plug-in classifier**
    - **NN and K-NN**
    - **Logistic Regression**
  - Clustering
    - **K-means**

Image Analysis - Motivation

Lund University

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