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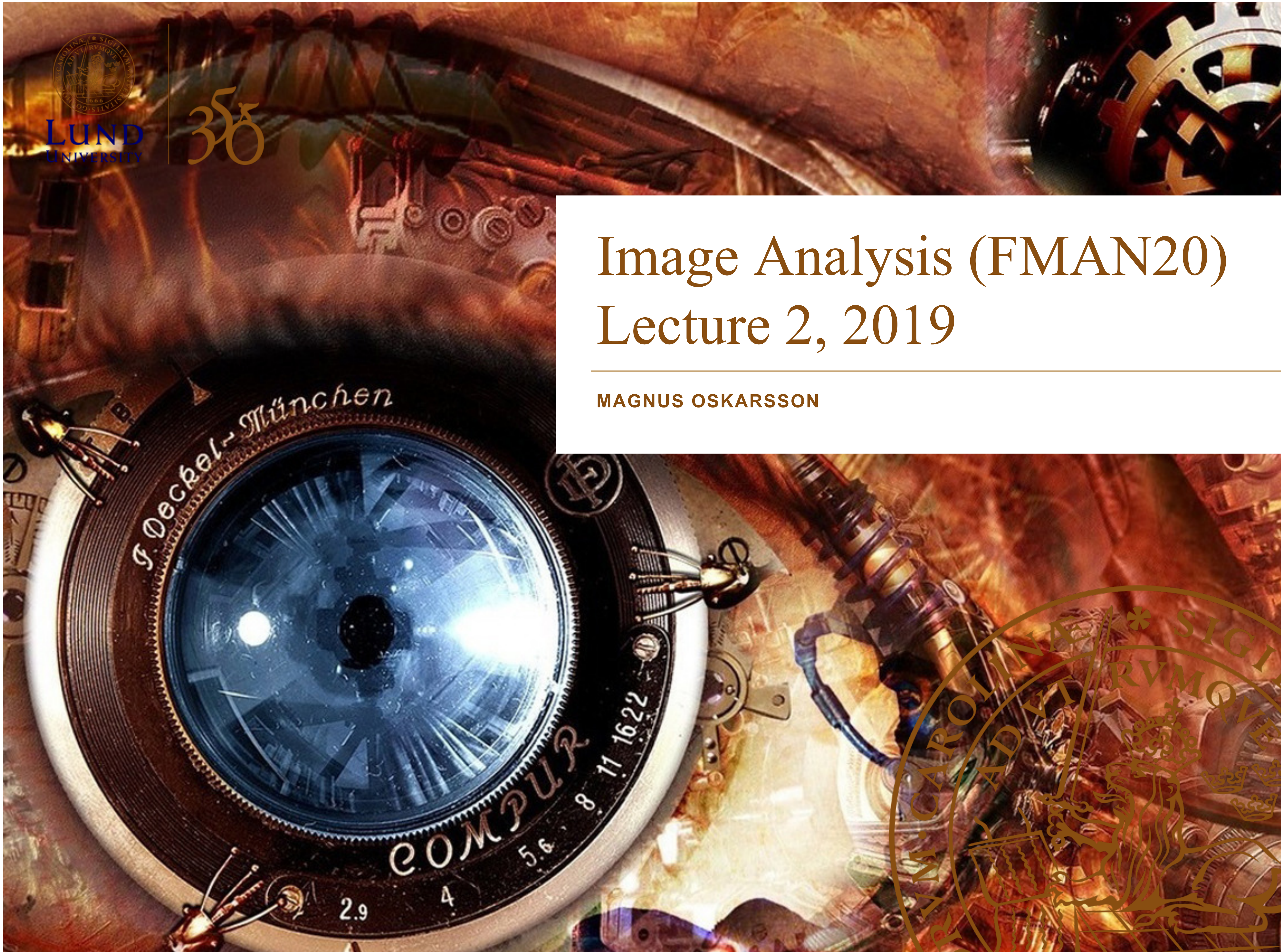
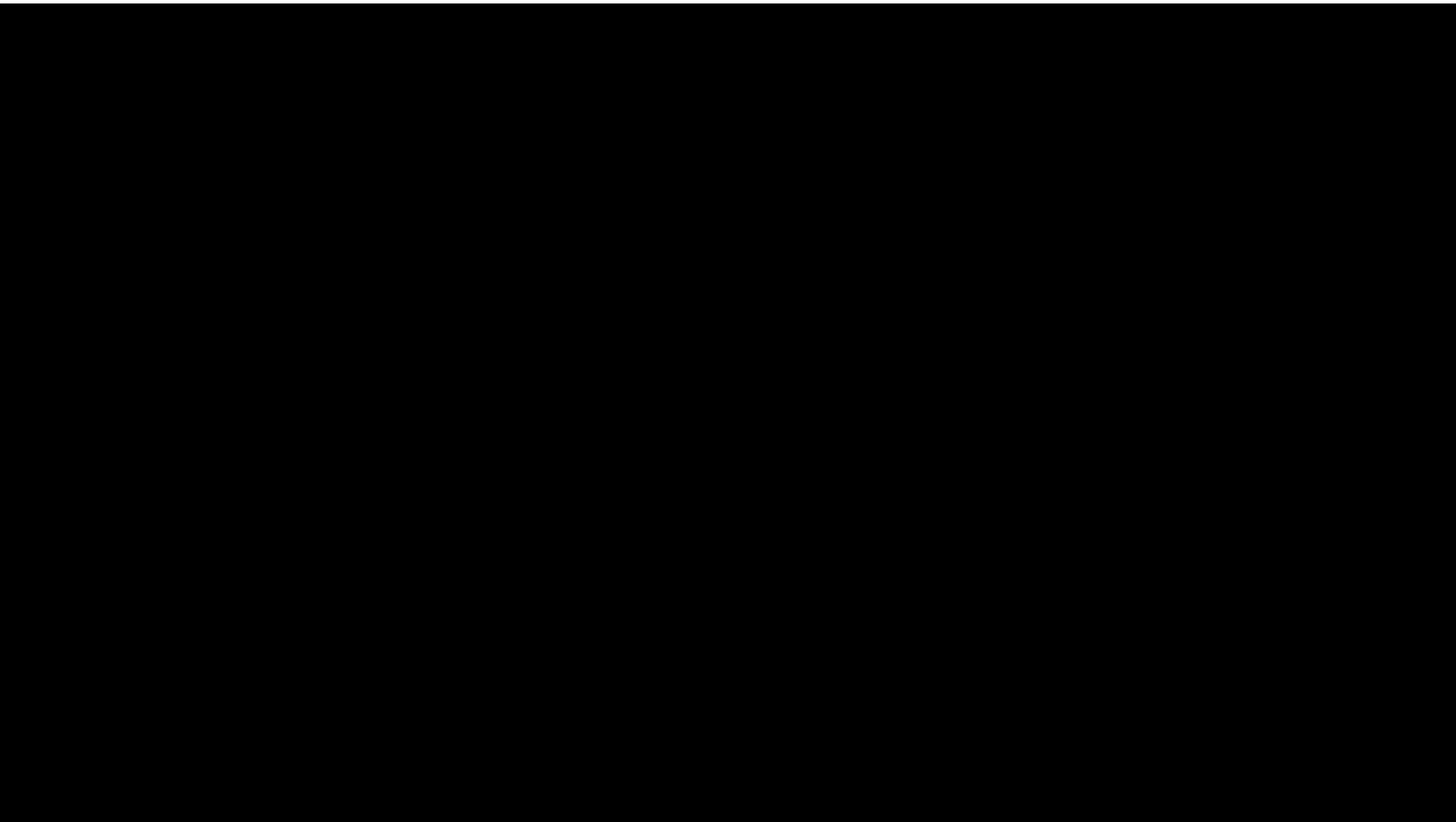


Image Analysis (FMAN20)
Lecture 2, 2019

MAGNUS OSKARSSON

Image Analysis - Motivation



Overview –
Linear Algebra and FFT

1. Linear Algebra
    1. **Vector space – ’A matrix is a vector’ What does this mean?**
    2. Basis, coordinates
    3. Scalar product
    4. Projection onto a subspace
    5. Projection onto an affine ‘subspace’
    6. (Principal Component Analysis – Recipe)
    7. Change of basis
2. Fourier Transform

But first, some notes on the home assignments….

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The Rules

Each student should hand in his or her own individual solution and should, upon request, be able to present the details in all the steps of the used algorithm. You are, however, allowed to discuss the assignment-problem with others. You may also ask your teachers and the course assistants for advice, if needed.

But first, some notes on the home assignments….

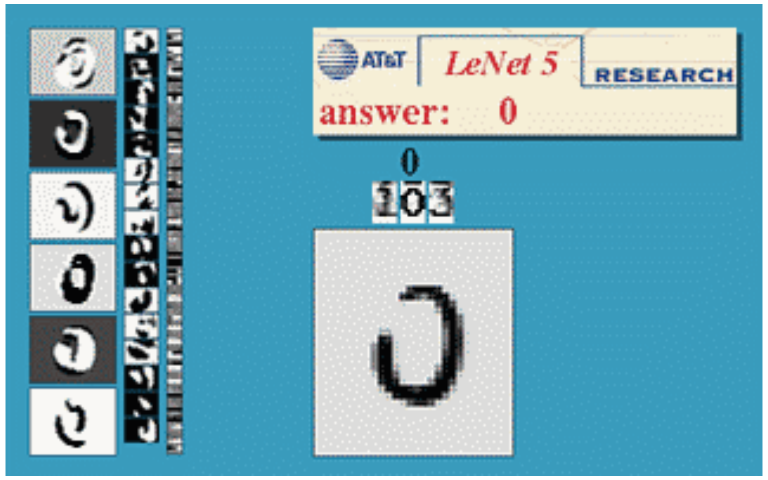
The report

Present your work in a report of approximately four A4-pages written in English or Swedish. Make sure you answer all questions in the grayed boxes and provide complete solutions to the exercises. The teacher is going to judge your work based on the report alone. Usually the teacher will check code only in very special cases, for instance if very persistent problems remain with your implementation. In these cases you may send code directly to the teacher that is correcting your assignment.

Examples of Classification problems

Examples of Classification problems

Optical character recognition (OCR)



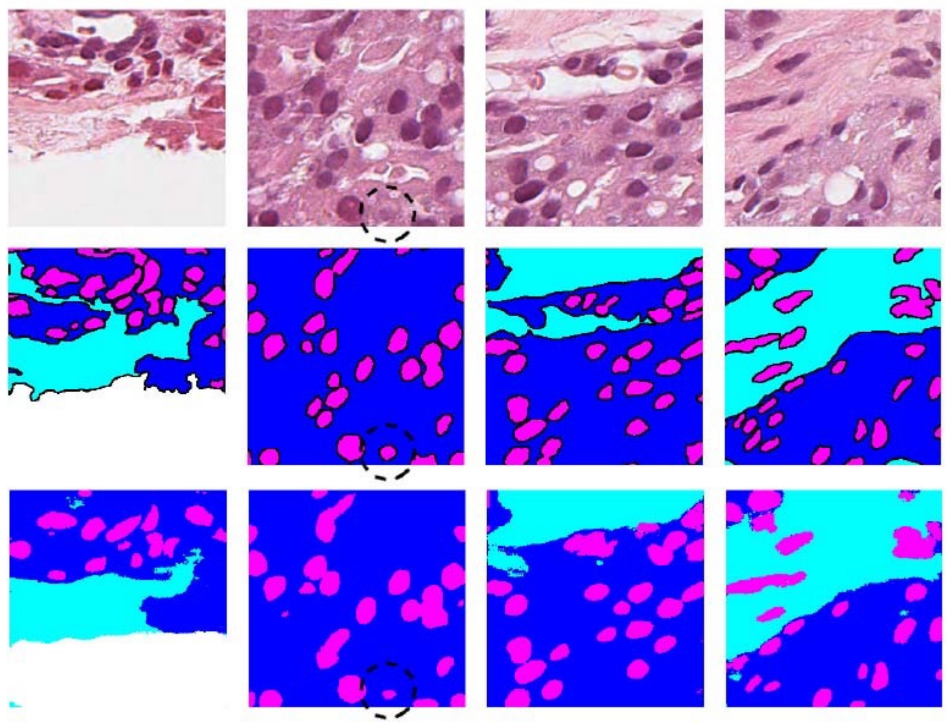
•Digit recognition, AT&T labs

•http://www.research.att.com/~yann/

•License plate readers

•http://en.wikipedia.org/wiki/Automatic_number_plate_recognition

Examples of Classification problems



Semantic Segmentation of Microscopic Images of H&E Stained Prostatic Tissue using CNN

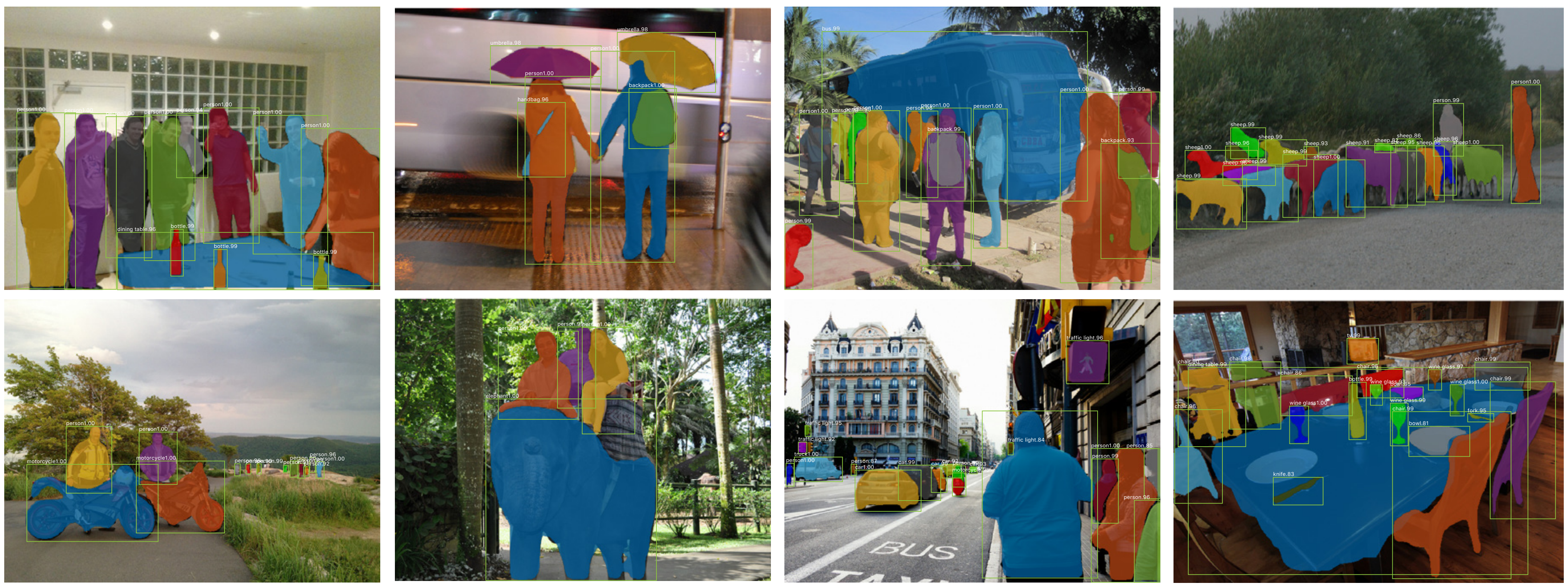
Johan Isaksson, Ida Arvidsson,
Kalle Åström and Anders Heyden
Lund University

Examples of Classification problems

Deep High-Resolution Representation Learning for Human Pose Estimation

Ke Sun[1,2*†] Bin Xiao[2*] Dong Liu[1] Jingdong Wang[2]
[1]University of Science and Technology of China [2]Microsoft Research Asia
{sunk,dongeliu}@ustc.edu.cn, {Bin.Xiao,jingdw}@microsoft.com

Examples of Classification problems



Mask R-CNN

Kaiming He Georgia Gkioxari Piotr Dollár Ross Girshick

Facebook AI Research (FAIR)

Machine Learning – classify

All of these classification problems have in common:

- data - $\mathbf{x}$ (after segmentation, extract features)
- A number of classes

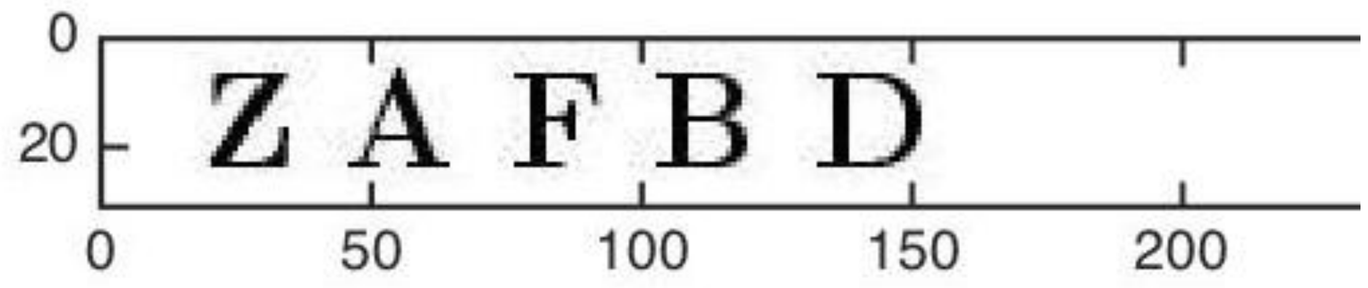
One would like to determine a class for every possible feature vector.
Here we will assume that the features are represented as a column vector, i.e. $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

One would like to compare the feature vector $\mathbf{x}$ with those that one usually gets with a number of classes. Let y denote the class index, i.e. the classes are $y \in \omega_y = \{1, \ldots, M\}$ where M denotes the number of classes.
Typical system: Image - filtering - segmentation - features - classification

Assignments: OCR project

Input: Image



Output: Text ’ZAFBD’

- Segmentation
- Features
- Classification
- Evaluation, benchmark

Vector spaces R^{n} and C^{n}

The following linear spaces are well-known:

- $\mathbb{R}^n$: all $n \times 1$-matrices, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in \mathbb{R}$
- $\mathbb{C}^n$: all $n \times 1$-matrices, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in \mathbb{C}$

Basis

Definition

$e_1, \ldots e_n \in \mathbb{R}^n$ is a **basis** in $\mathbb{R}^n$ if

- they are linearly independent
- they span $\mathbb{R}^n$.

■

Example (3D space)

$e_1, e_2, e_3 \in \mathbb{R}^3$ is a **basis** *in $\mathbb{R}^3$ if they are not located in the same plane.* ■

Canonical basis (normal basis)

Example (canonical basis)

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

is called the **canonical basis** *in* $\mathbb{R}^n$*.*

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 \dots + x_n e_n \ .$$

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Coordinates

Let $e_1, e_2, \ldots, e_n$ be a basis. Then for every x there is a unique set of scalars ξ_i such that

$$x = \sum_{i=1}^{n} \xi_i e_i \quad .$$

These scalars are called the **coordinates** for x in the basis $e_1, e_2, \ldots, e_n$.

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Scalar product

Definition

Let A be a (complex) matrix. Introduce

$$A^* = (\bar{A})^T \quad .$$

■

Definition

Let x and y be two vectors in $\mathbb{R}^n$ ($\mathbb{C}^n$). **The scalar product** of x and y is defined as

$$x \cdot y = \sum \bar{x}_i y_i = x^* y \quad .$$

■

General Vector Space

- A ’General’ Vector Space is a collection of objects called **vectors**, which can be added together and also be multiplied by ’numbers’ called **scalars**, where the **addition** and **multiplication with scalars** fulfill a set of rules.

(i)	$\bar{\mathbf{u}} + \bar{\mathbf{v}} = \bar{\mathbf{v}} + \bar{\mathbf{u}}$	*(commutativity)*
(ii)	$(\bar{\mathbf{u}} + \bar{\mathbf{v}}) + \bar{\mathbf{w}} = \bar{\mathbf{u}} + (\bar{\mathbf{v}} + \bar{\mathbf{w}})$	*(associativity)*
(iii)	$\bar{\mathbf{v}} + \bar{\mathbf{0}} = \bar{\mathbf{v}}$	*(zero existence)*
(iv)	$\bar{\mathbf{v}} + (-\bar{\mathbf{v}}) = \bar{\mathbf{0}}$	*(negative vector existence)*
(v)	$k(l\bar{\mathbf{v}}) = (kl)\bar{\mathbf{v}}$	*(associativity)*
(vi)	$1\bar{\mathbf{v}} = \bar{\mathbf{v}}$	*(multiplicative one)*
(vii)	$0\bar{\mathbf{v}} = \bar{\mathbf{0}}$	*(multiplicative zero)*
$(viii)$	$k\bar{\mathbf{0}} = \bar{\mathbf{0}}$	*(multiplicative zero vector)*
(ix)	$k(\bar{\mathbf{u}} + \bar{\mathbf{v}}) = k\bar{\mathbf{u}} + k\bar{\mathbf{v}}$	*(distributivity 1)*
(x)	$(k + l)\bar{\mathbf{v}} = k\bar{\mathbf{v}} + l\bar{\mathbf{v}}$	*(distributivity 2)*

General Vector Space

- A ’General’ Vector Space is a collection of objects called **vectors**, which can be added together and also be multiplied by ’numbers’ called **scalars**, where the **addition** and **multiplication with scalars** fulfill a set of rules.
- There are many examples of such vectors spaces. The vectors can for example be
  - Geometrical vectors in three dimensions
  - N-tuples of real numbers
  - Functions
  - Polynomials
  - Matrices
  - Tensors

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Example - polynomials

- Vectors - Polynomials of degree 2
- Scalars – Real numbers

Example 3.2.1. *Polynomials in one variable of degree 2 is a vector space. One possible basis is*

$$\bar{\mathbf{e}}_1(x) = 1, \quad \bar{\mathbf{e}}_2(x) = x, \quad \bar{\mathbf{e}}_3(x) = x^2.$$

The polynomial $\bar{\mathbf{u}}(x) = 5x^2 + 3x - 2$ *has coordinates* $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$, *since*

$$\bar{\mathbf{u}} = \underbrace{u_1}_{-2}\underbrace{\bar{\mathbf{e}}_1}_{1} + \underbrace{u_2}_{3}\underbrace{\bar{\mathbf{e}}_2}_{x} + \underbrace{u_3}_{5}\underbrace{\bar{\mathbf{e}}_3}_{x^2} = 5x^2 + 3x - 2.$$

The dimension of the vector space is 3.

Example - matrices

- Vectors – Matrices of size 2x2
- Scalars – Real numbers

Example 3.2.2. *Matrices of size* 2×2 *is a vector space. One possible basis is*

$$\bar{\mathbf{e}}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\mathbf{e}}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \bar{\mathbf{e}}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{\mathbf{e}}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix

$$\bar{\mathbf{u}} = \begin{pmatrix} 1 & 7 \\ 3 & 2 \end{pmatrix}$$

has coordinates $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 7 \\ 2 \end{pmatrix}$, *since*

$$\bar{\mathbf{u}} = \underbrace{u_1}_{1} \underbrace{\bar{\mathbf{e}}_1}_{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} + \underbrace{u_2}_{3} \underbrace{\bar{\mathbf{e}}_2}_{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} + \underbrace{u_3}_{7} \underbrace{\bar{\mathbf{e}}_3}_{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} + \underbrace{u_4}_{2} \underbrace{\bar{\mathbf{e}}_4}_{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 7 \\ 3 & 2 \end{pmatrix}$$

The dimension of the vector space is 4.

Image matrix

$$f = \begin{bmatrix} f(1,1) & f(1,2) & \dots & f(1,N) \\ f(2,1) & f(2,2) & \dots & f(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ f(M,1) & f(M,2) & \dots & f(M,N) \end{bmatrix}$$

$$f(j,\cdot) = [f(j,1)\ f(j,2)\ \dots\ f(j,N)] \ ,$$

$$f(\cdot,k) = \begin{bmatrix} f(1,k) \\ f(2,k) \\ \vdots \\ f(M,k) \end{bmatrix} \ .$$

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Column stacking

$$\widetilde{f} = \begin{bmatrix} f(\cdot,1) \\ f(\cdot,2) \\ \vdots \\ f(\cdot,N)\,. \end{bmatrix}$$

$$\widetilde{f+g} = \widetilde{f} + \widetilde{g}, \quad \widetilde{\lambda f} = \lambda \widetilde{f}$$

Set of images is a vector space

- Images are a vector space (with scalar product)
  - Addition
  - Multiplication by scalar
- Two ways to think of 'images' as vectors (both are the same)
- 1. Column stacking
  - Use column stacking to convert to 'old school' vector R^n
  - Use linear algebra as usual
  - Convert back to matrix form when needed
- 2. Image basis
  - Choose a basis (any basis).
  - Through the use of coordinates, obtain vector representation
  - Use linear algebra as usual
  - Convert back when needed

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Canonical basis

$$\chi(i,j) = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ & \vdots & 1 & \vdots & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix},$$

with the 1 at position (i,j).
Using this canonical basis we can write

$$f = \sum_{i,j} f(i,j)\chi(i,j) \ .$$

Idea for image transform:
Choose another basis that is more suitable in some sense.
Image matrices can thus be seen as vectors in a linear space.

Scalar product of images

Definition

The scalar product of two matrices (images) is defined as

$$f \cdot g = \sum_{i=1}^{M} \sum_{j=1}^{N} \bar{f}(i,j) g(i,j) \quad .$$

$x, y \in \mathbb{R}(\mathbb{C})$ are **orthogonal** if $x \cdot y = 0$. This is often written

$$x \perp y \quad \Leftrightarrow \quad x \cdot y = 0 \quad .$$

The length or **the norm** of x is defined as

$$||f|| = \sqrt{f \cdot f} = \sqrt{\sum_{i=1}^{M} \sum_{j=1}^{N} \bar{f}(i,j) f(i,j)}.$$

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Scalar product and norm

Example 3.2.1 (Scalar product and norm). *Let*

$$f = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$$

and

$$g = \begin{pmatrix} 4 & 2 \\ -1 & -3 \end{pmatrix}.$$

What is the scalar product $f \cdot g$*? What is the norm* $||f||$*?*

$$f \cdot g = \sum_{i=1}^{M} \sum_{j=1}^{N} \bar{f}(i,j) g(i,j). \quad ||f|| = \sqrt{f \cdot f} = \sqrt{\sum_{i=1}^{M} \sum_{j=1}^{N} \bar{f}(i,j) f(i,j)}.$$

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Orthonormal basis

Definition

$\{e_1, \ldots, e_n\}$ is an **orthonormal (ON-) basis** in $\mathbb{R}^n$ ($\mathbb{C}^n$) if

$$e_i \cdot e_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

■

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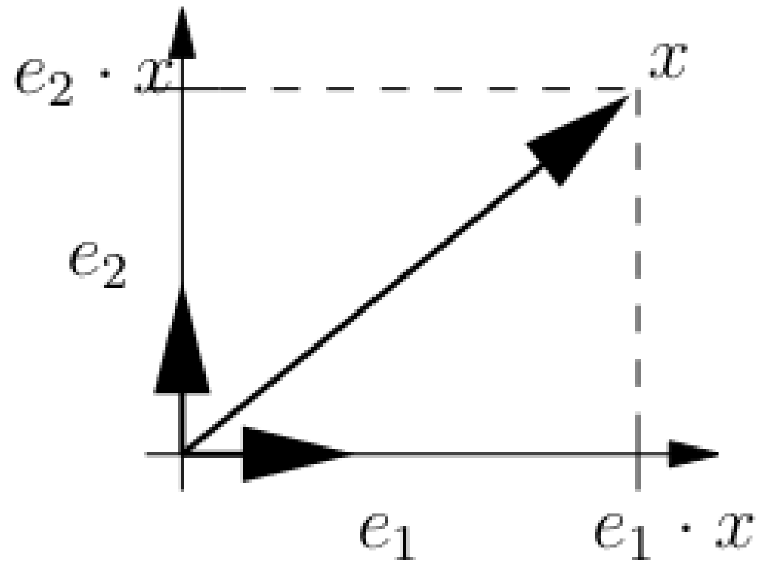
Orthonormal basis



Theorem

Assume that $\{e_1, \ldots, e_n\}$ *is orthonormal (ON) basis and*

$$x = \sum_{i=1}^{n} \xi_i e_i \quad .$$

Then

$$\xi_i = e_i \cdot x = e_i^* x, \quad ||x||^2 = \sum_{i=1}^{n} |\xi_i|^2$$

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Orthogonal projection

Definition

Let $\{a_1, \ldots, a_k\} \in \mathbb{R}^n$, $k \leq n$, span a linear subspace, π, in $\mathbb{R}^n$, i.e.:

$$\pi = \{w | w = \sum_{i=1}^{k} x_i a_i,\ x_i \in \mathbb{R}\} \ .$$

The **orthogonal projection** of $u \in \mathbb{R}^n$ on π is a linear mapping P, such that $u_\pi = Pu$ and defined by

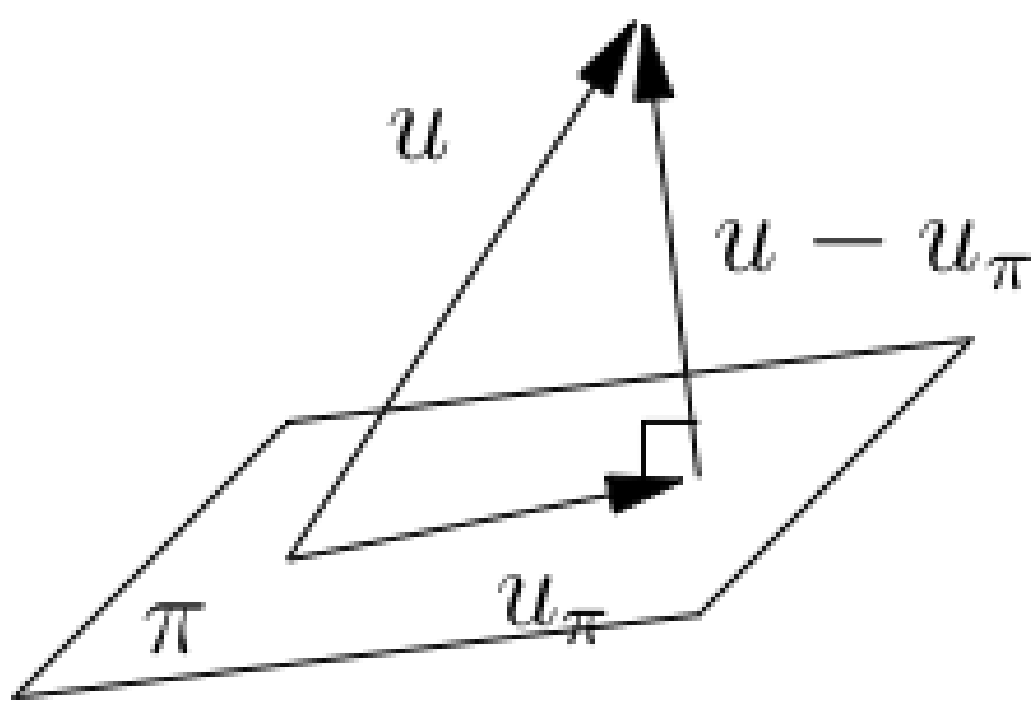
$$\min_{w \in \pi} ||u - w|| = ||u - u_\pi|| \ .$$

■

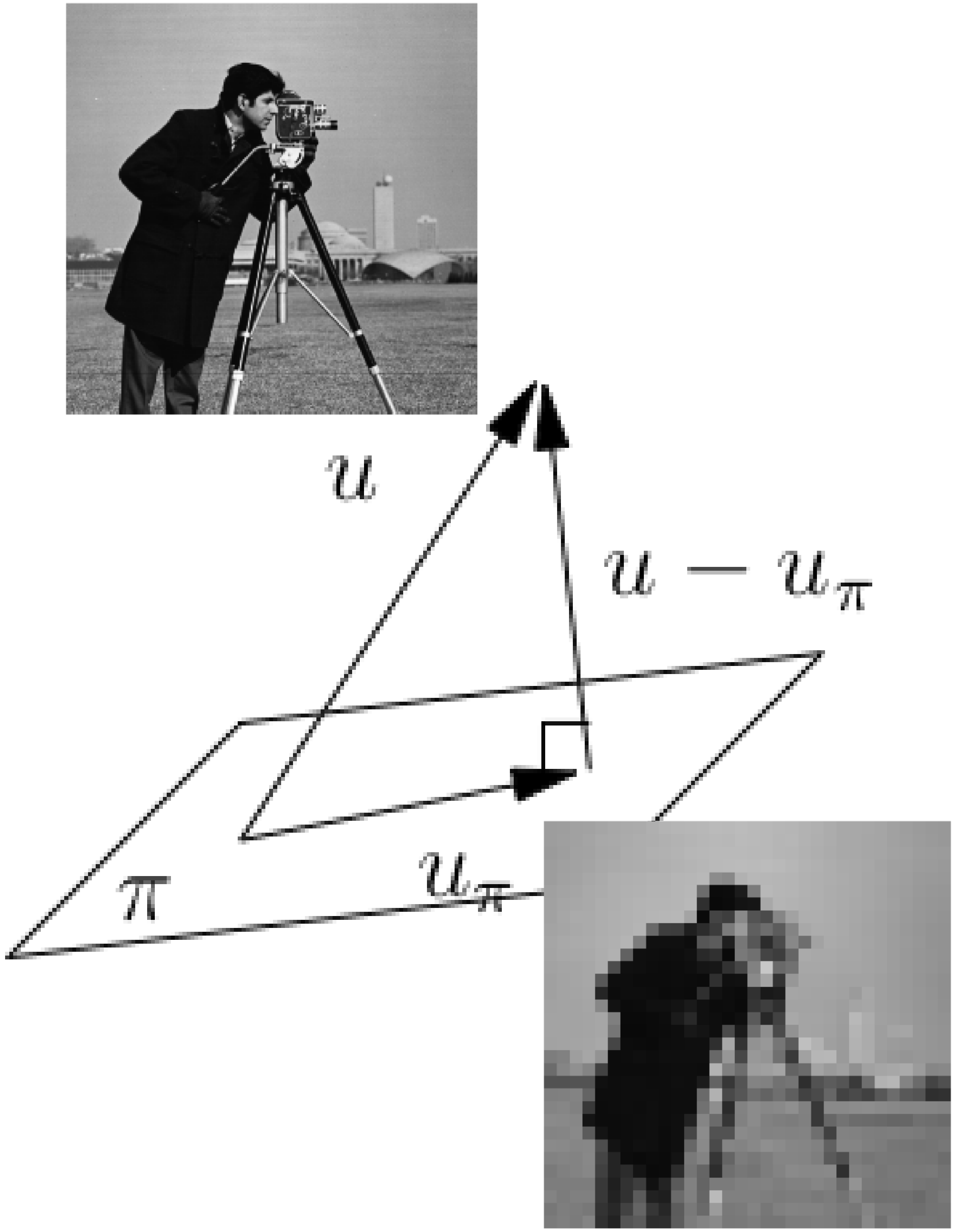
Orthogonal projection

The orthogonal projection is characterized by

1. $u_\pi \in \pi$
2. $u - u_\pi \perp w$ for every $w \in \pi$



Orthogonal projection



Let $a \in \pi$ and $b \in \pi$ be two solutions to the minimisation problem. Set

$$f(t) = \|u - ta - (1-t)b\|^2 = \dots$$

$$= \|u - b\|^2 + t^2\|a - b\|^2 - 2t(a - b) \cdot (u - b), \quad t \in \mathbb{R}\,.$$

This is a second degree polynomial with minimum in $t = 0$ and $t = 1 \Rightarrow f(t)$ is a constant function and thus $\Rightarrow a = b$.

Let $f(t) = \|u - u_\pi + ta\|^2$, where $a \in \pi$. It follows that $f'(0) = 2(u - u_\pi) \cdot a = 0$, i.e. $(u - u_\pi) \perp a$.
Conversely: Assume $w \in \pi$. The property that $(u - u_\pi) \perp a$, for every $a \in \pi$ gives that

$$\|u - w\|^2 = \|u - u_\pi + u_\pi - w\|^2 =$$

$$\|u - u_\pi\|^2 + \|u_\pi - w\|^2 \geq \|u - u_\pi\|^2,$$

i.e. u_π solves the minimization problem.

Let $A = [a_1 \dots a_k]$ be a $n \times k$ matrix and

$$\pi = \{w | w = Ax,\ x_i \in \mathbb{R}^n\}$$

Lemma

If $\{a_1, \dots, a_k\}$ are linearly independent $\mathbb{R}^n$ then $A^* A$ is invertible.

Proof: Do it on your own. (Use SVD if you are familiar with it.)

■

Theorem

if the columns of A are linearly independent, then the projection of u on π is given by

$$u_\pi = x_1 a_1 + \ldots + x_k a_k, \quad x = (A^* A)^{-1} A^* u \ .$$

Proof: Use the characterization of the projection (above).

$$a_i^*(u - u_\pi) = 0 \quad \Rightarrow$$

$$A^*(u - Ax) = 0 \quad \Rightarrow$$

$$A^* u = A^* A x \quad \Rightarrow \quad x = (A^* A)^{-1} A^* u$$

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Definition

$A^+ = (A^*A)^{-1}A^*$ is called the **pseudo-inverse** of A. ■

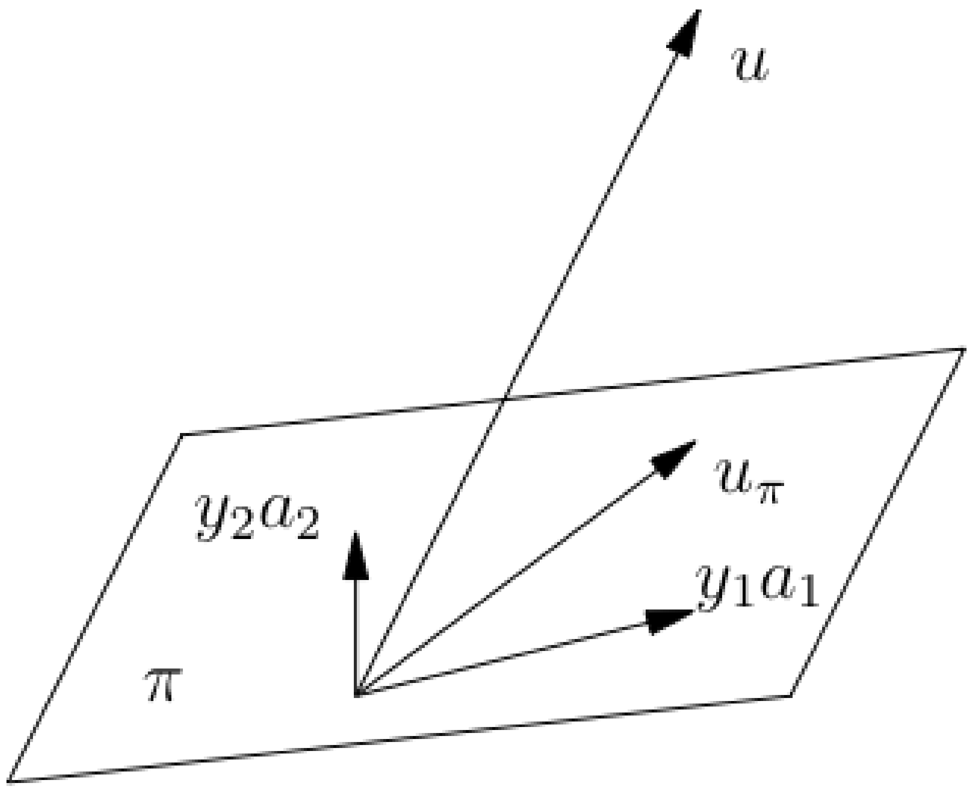
Observe that if A is quadratic and invertible then $A^+ = A^{-1}$.

Theorem

If $\{a_1, \ldots, a_k\}$ are orthonormal, then the projection of u on π is given by

$$u_\pi = y_1 a_1 + \ldots + y_k a_k, \quad y_i = a_i^* u \ .$$

Proof: This follows from $A^*A = I$. ■



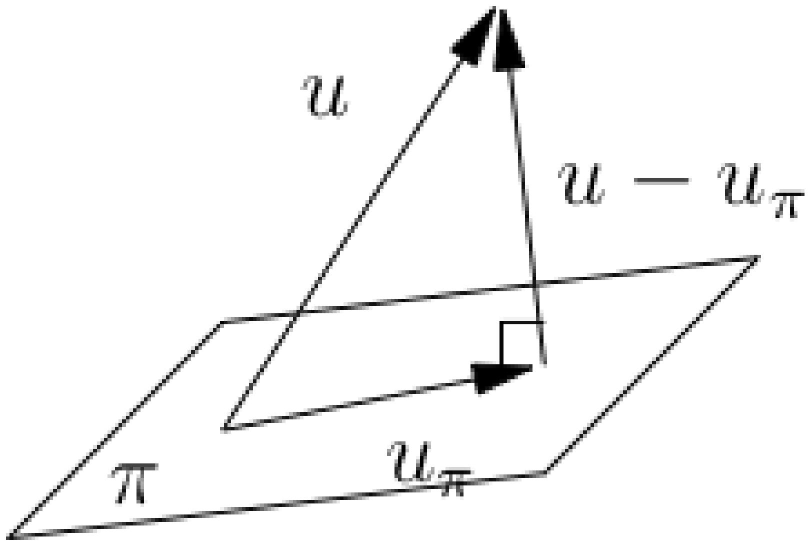
Orthogonal projection

What is the orthogonal projection of f

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 7 \end{pmatrix}$$

onto the space spanned by (e_1, e_2, e_3)

$$e_1 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}, e_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$



$$f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 7 \end{pmatrix}$$

Orthogonal projection

$$e_1 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}, e_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

Since (e_1, e_2, e_3) is orthonormal the coordinates are
$x_1 = f \cdot e_1 = 14, x_2 = f \cdot e_2 = -15/\sqrt{6}, x_3 = f \cdot e_3 = -4/\sqrt{6}$.
The orthogonal projection is then
$\hat{f} = 14e_1 - 15/\sqrt{6}e_2 - 4/\sqrt{6}e_3$

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 7 \end{pmatrix}, \hat{f} = \begin{pmatrix} 1.5 & 2\frac{1}{6} & 2\frac{5}{6} \\ 4 & 4\frac{2}{3} & 5\frac{1}{3} \\ 6.5 & 7\frac{1}{6} & 7\frac{5}{6} \end{pmatrix},$$

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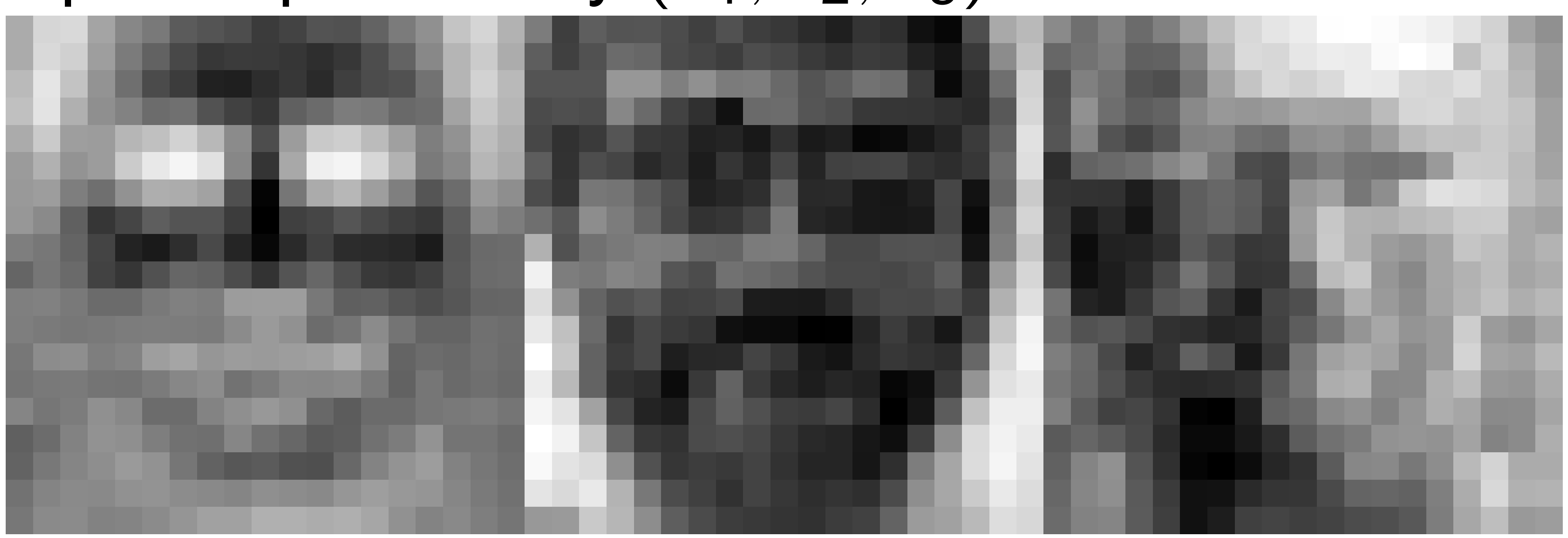
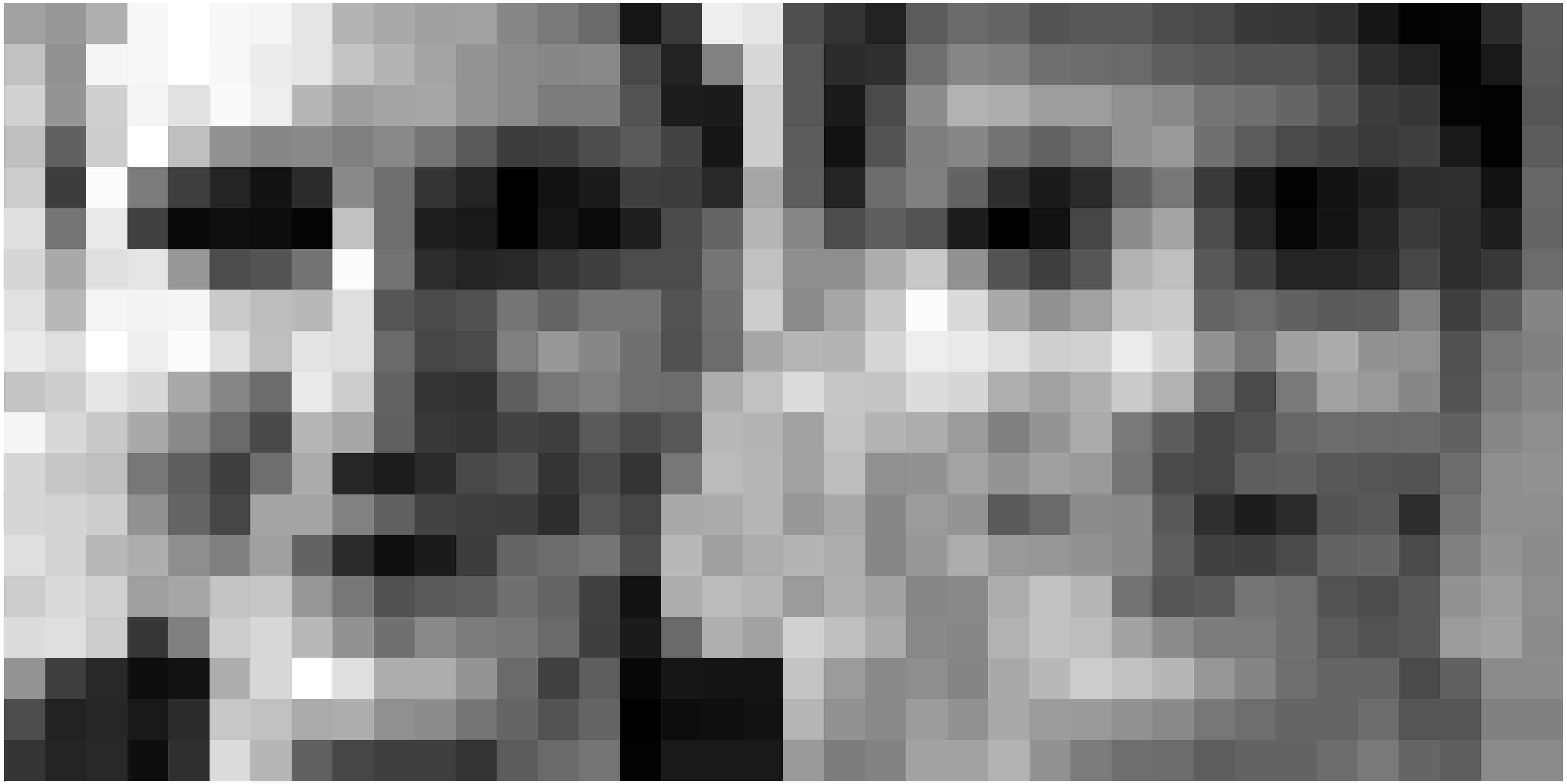
X

What is the orthogonal projection of f

onto the space spanned by (e_1, e_2, e_3)



Since (e_1, e_2, e_3) is orthonormal, the coordinates are
$x_1 = f \cdot e_1 = -2457, x_2 = f \cdot e_2 = 303, x_3 = f \cdot e_3 = -603.$
The orthogonal projection is then $\hat{f} = -2457e_1 + 303e_2 - 603e_3$



Overview – Linear Algebra and FFT

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Projection onto affine subspace

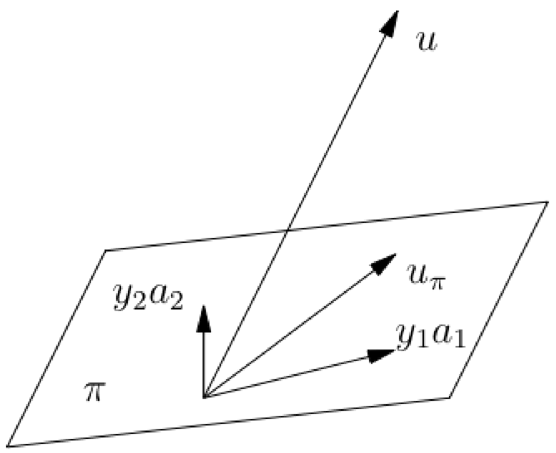
- Previously projection onto linear subspace

$$\pi = \{w \mid w = \sum_1^n x_i a_i = Ax \quad \text{where} \quad x_i \in \mathbb{C}\ (\text{or}\mathbb{R})\}$$

- A linear subspace always contains the zero vector
- How about planes or 'subspaces' that are shifted away from the origin. Such sets are called affine spaces.

$$\pi = \{w \mid w = m + \sum_1^n x_i a_i = Ax + m \quad \text{where} \quad x_i \in \mathbb{C}\ (\text{or}\mathbb{R})\}.$$

- An affine subspace is typically not a linear space



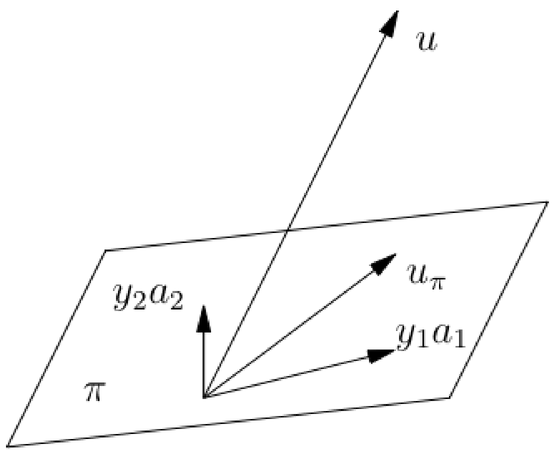
Projection onto affine subspace

- An affine subspace, defined by $m, a_1, \ldots, a_k$.

$$\pi = \{w \mid w = m + \sum_1^n x_i a_i = Ax + m \quad \text{where} \quad x_i \in \mathbb{C}\ (\text{or}\mathbb{R})\}.$$

- Projection of u onto the affine subspace

  - Subtract m, i.e. form $v = u - m$.
  - Project v onto the space spanned by $a_1, \ldots, a_k$, i.e. $v_\pi = A^+ v$.
  - Add m, i.e. form $u_\pi = v_\pi + m$.



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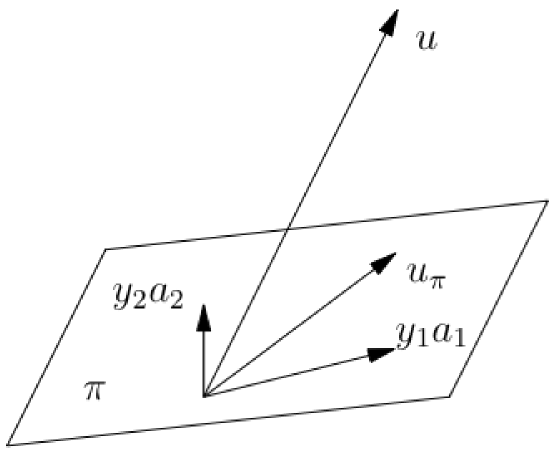
PCA - Principal Component Analysis

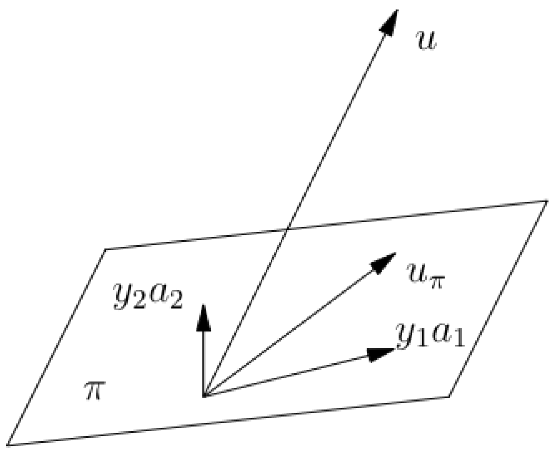


- Orthogonal projection – project an image u on
  - subspace spanned by $a_1, \ldots, a_k$.
  - or affine subspace defined by $m, a_1, \ldots, a_k$.

- How do we find a good subspace?
- Given lots of vectors $x_1, \ldots, x_N$. Find a suitable affine subspace so that the orthogonal projections y_i of x_i are as close to x_i as possible

$$e(\pi) = \sum_{i=1}^{N} ||y_i(\pi) - x_i||^2.$$

PCA - Principal Component Analysis

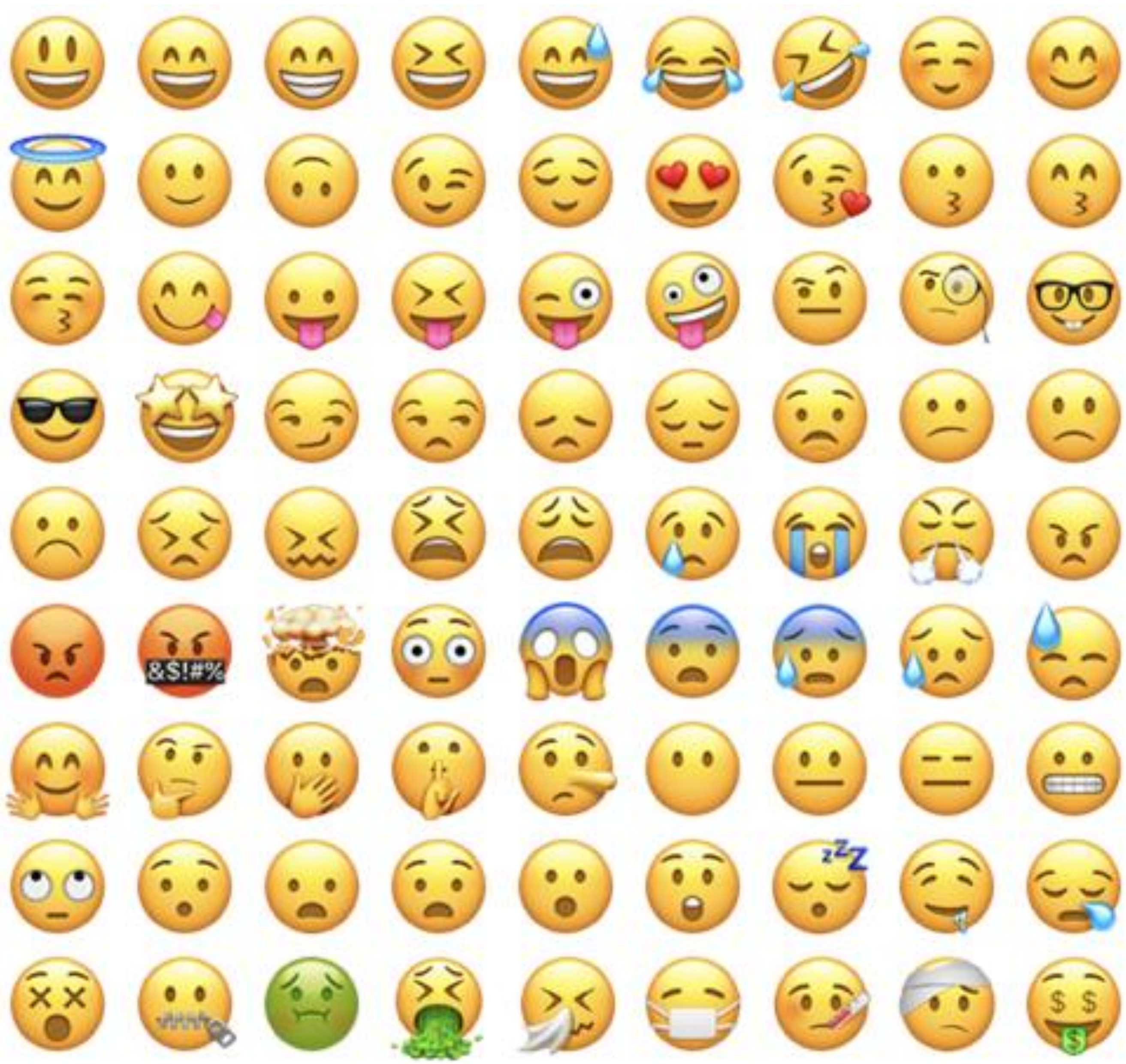


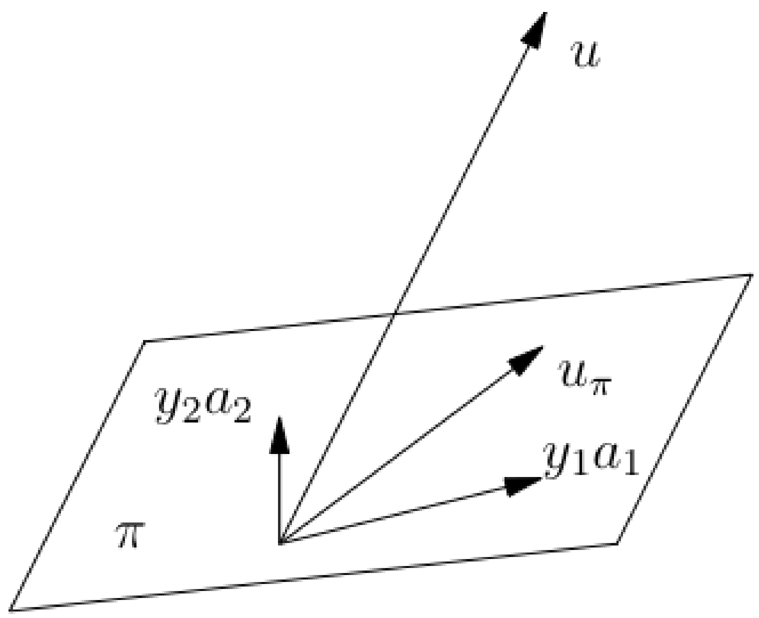
1. Calculate the mean $m = \frac{1}{N}\sum_{i=1}^{N} x_i$.
2. Subtract the mean from all examples $z_i = x_i - m$.
3. Place all of the resulting vectors as columns of a matrix, $M = \begin{pmatrix} z_1 & \dots & z_N \end{pmatrix}$.
4. Factorize M using the singular value decomposition $M = USV^T$.
5. Use the first k columns of U as the basis of the subspace, i.e. $a_i = u_i$, with $U = \begin{pmatrix} u_1 & \dots & u_m \end{pmatrix}$.

$$\pi = \{w \mid w = m + \sum_{1}^{n} x_i a_i = Ax + m \quad \text{where} \quad x_i \in \mathbb{C}\ (\text{or}\mathbb{R})\}.$$

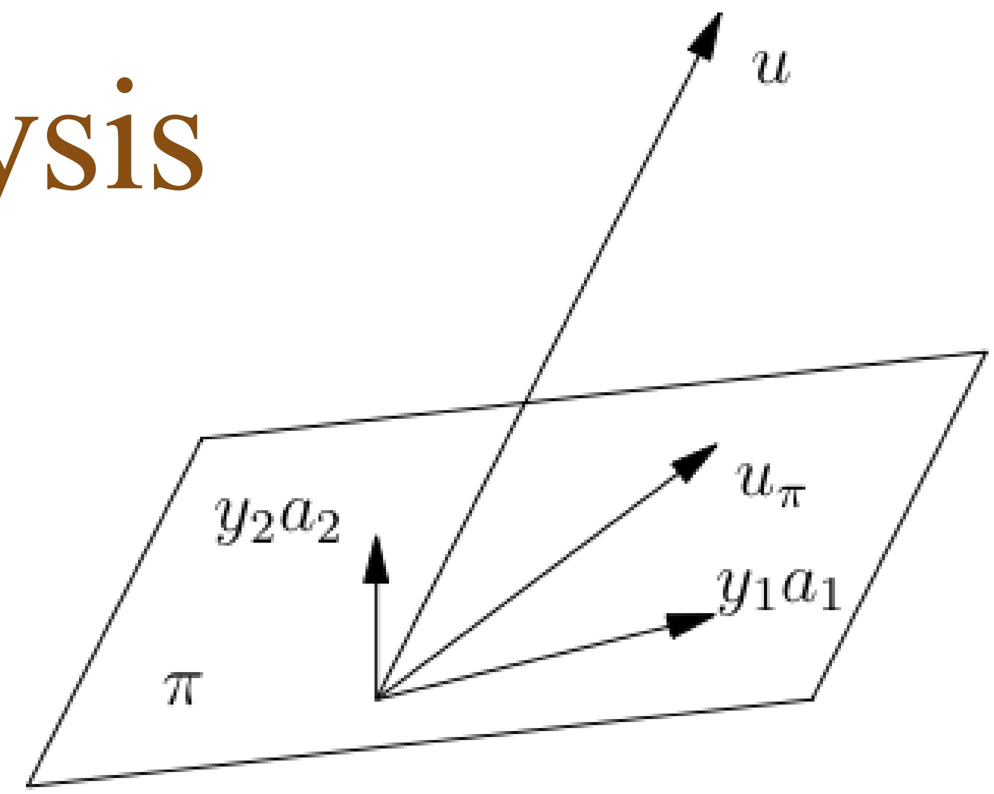
$$e(\pi) = \sum_{i=1}^{N} ||y_i(\pi) - x_i||^2.$$

PCA –"Training"
Given examples, find subspace





PCA - Principal Component Analysis

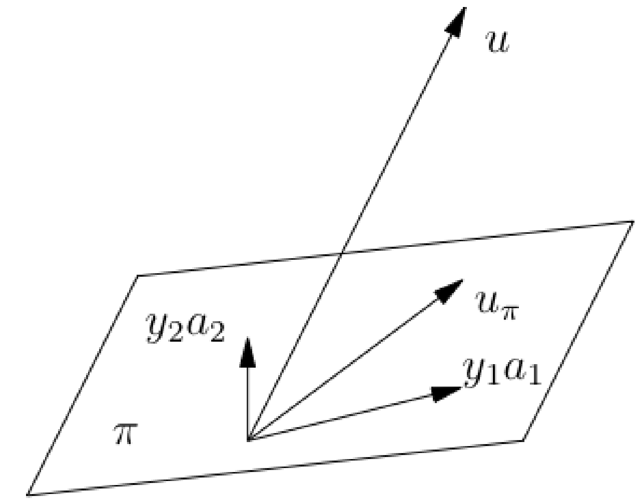



Mean Emoji


Eigen-Emoji a_1


Eigen-Emoji a_2

PCA - Principal Component Analysis

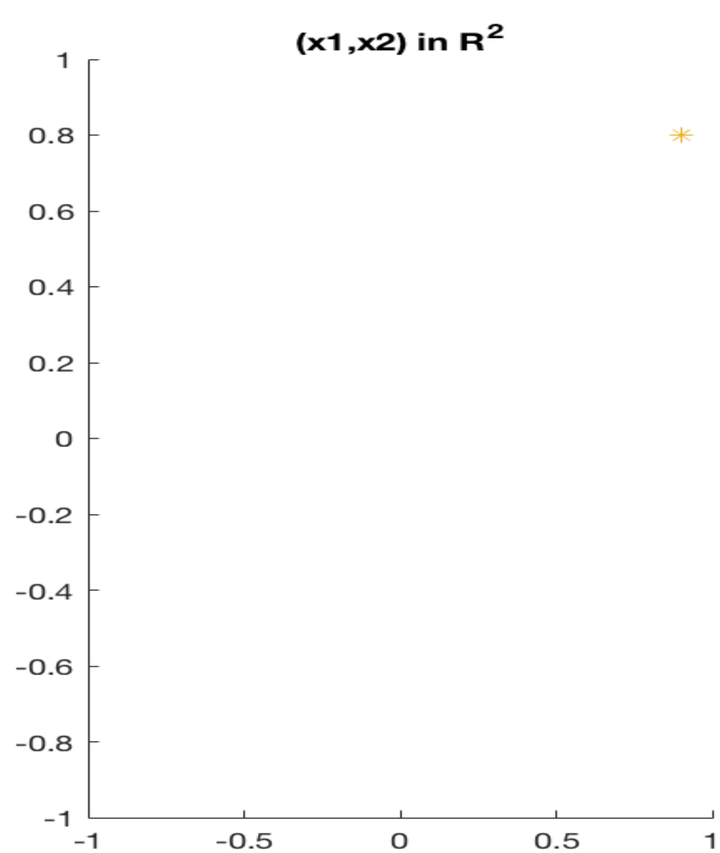


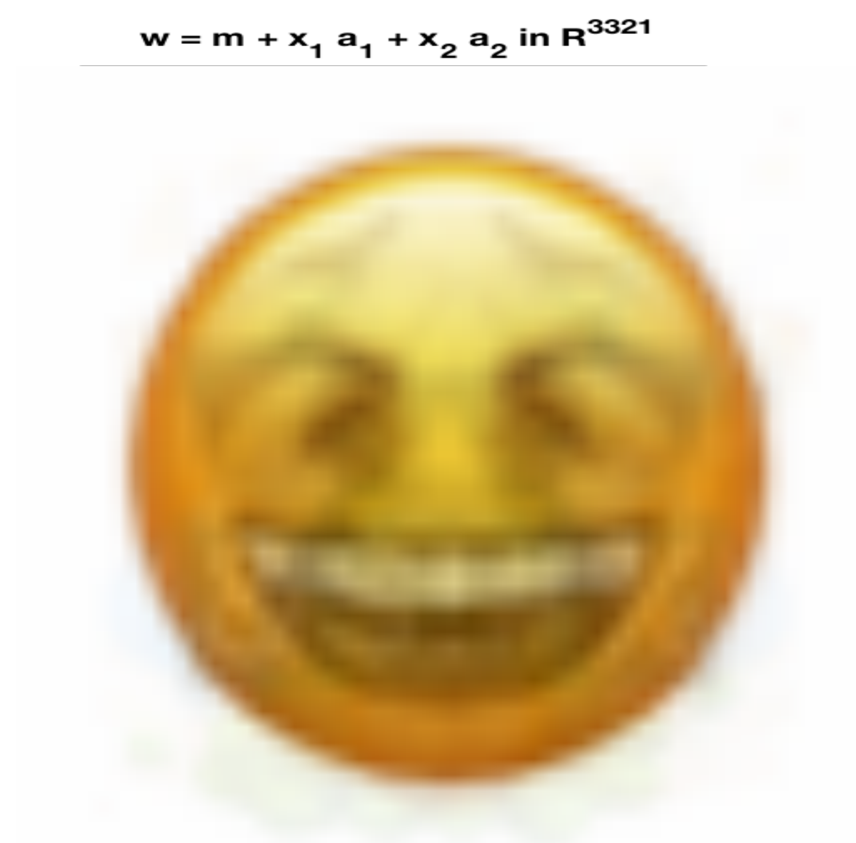

Mean Emoji


Eigen-Emoji a_1

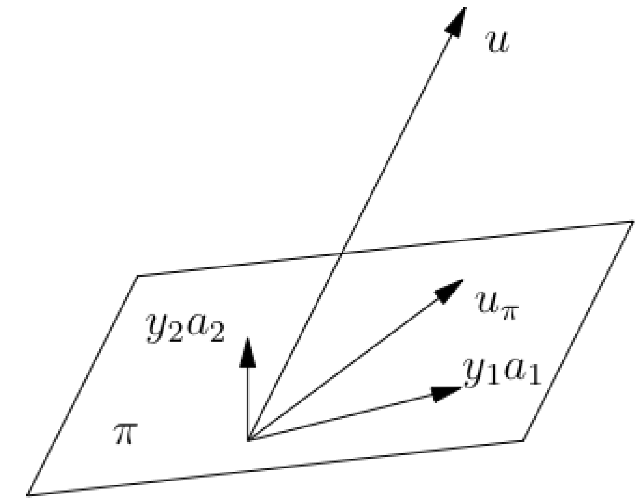

Eigen-Emoji a_2

$$w = m + \sum_{1}^{n} x_i a_i$$





PCA - Principal Component Analysis



Mean Emoji

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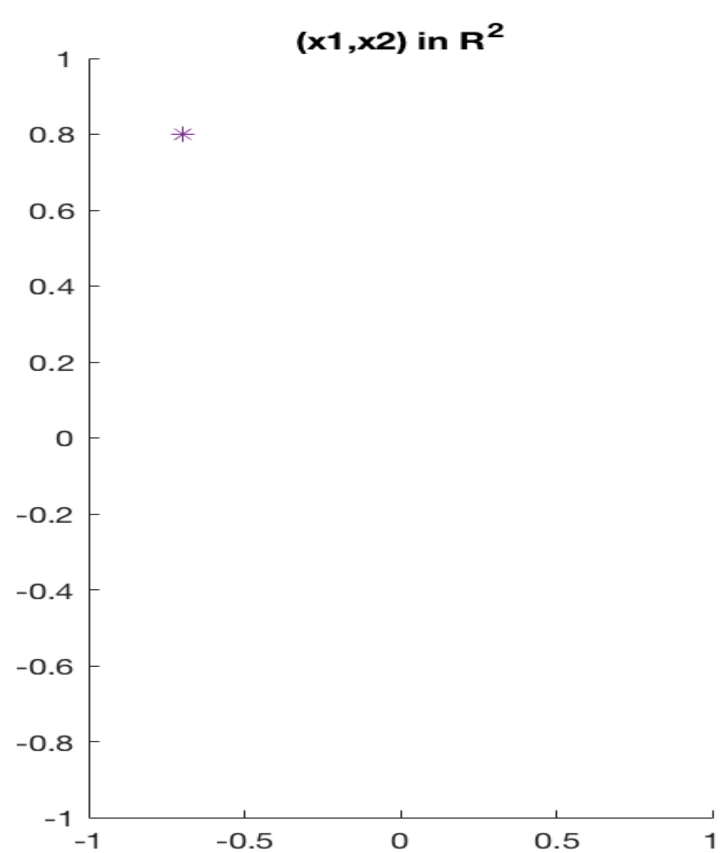
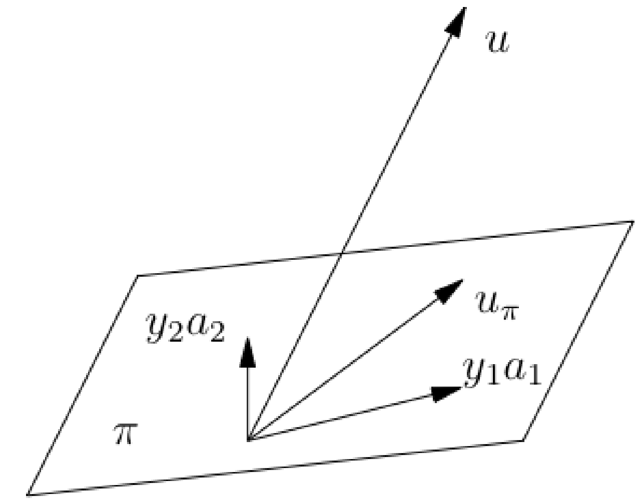
$$w = m + \sum_{1}^{n} x_i a_i$$



PCA - Principal Component Analysis

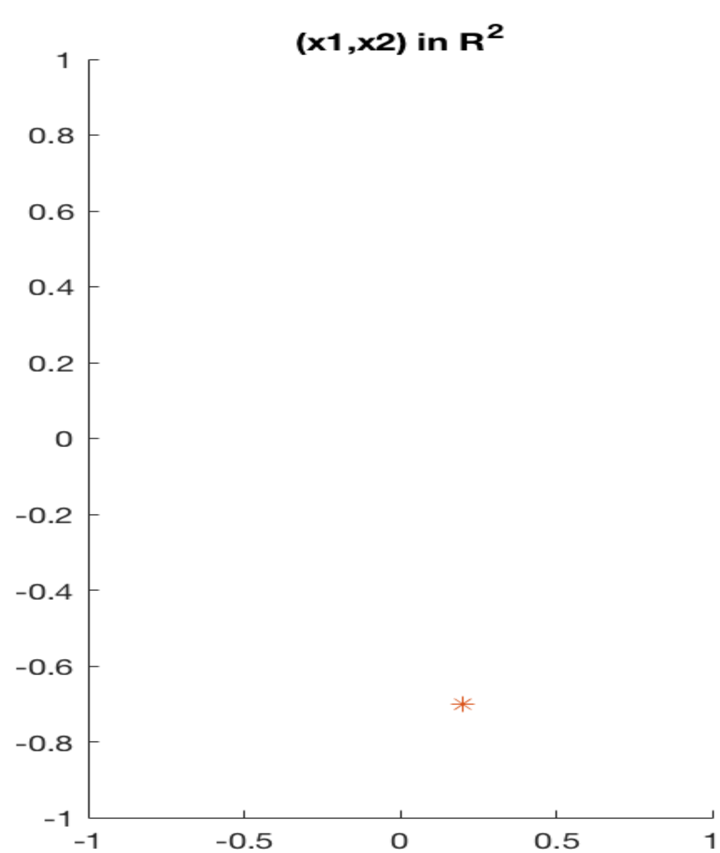


$$w = m + \sum_{1}^{n} x_i a_i$$

Mean Emoji

Eigen-Emoji a_1

Eigen-Emoji a_2



PCA - Principal Component Analysis

Approximation of new shapes using PCA basis elements

New shape not in database

using 10 coefficients

using 50 coefficients

using 100 coefficients

using 500 coefficients

Overview – Linear Algebra and FFT

1. Linear Algebra
   1. Vector space – ’A matrix is a vector’ What does this mean?
   2. Basis, coordinates
   3. Scalar product
   4. Projection onto a subspace
   5. Projection onto an affine ‘subspace’
   6. (Principal Component Analysis – Recipe)
   7. Change of basis
2. **Fourier Transform**

Fourier Transform

$$F(u,v) = \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y)\, e^{-i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)}$$

- Can be viewed as a change of basis
- Image f –> Fourier Transform F   (and back)
- Has strong connections with convolutions
- (next lecture)
- Useful for image compression
- Useful for image understanding
- Basically a great tool

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Fourier Transform

- Definition, is a change of basis, what does is mean
  - Detour (for increased understanding
  - Ordinary Fourier Transform (from previous courses)
  - Examples
  - Properties
- Discrete Fourier Transform – 1D

Image basis example (Walsh)

$$f = \begin{bmatrix} 9 & -1 \\ 5 & 7 \end{bmatrix}$$

$$\Phi_{11} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}/2 \quad \Phi_{12} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}/2$$

$$\Phi_{21} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}/2 \quad \Phi_{22} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}/2$$

$$\boxed{\begin{gathered} x_{ij} = f \cdot \Phi_{ij} = \sum_{\lambda,\mu} f(\lambda,\mu)\Phi_{ij}(\lambda,\mu) \\ f = x_{11}\Phi_{11} + x_{21}\Phi_{21} + x_{12}\Phi_{12} + x_{22}\Phi_{22} \end{gathered}}$$

$$x = \begin{bmatrix} 10 & 4 \\ -2 & 6 \end{bmatrix}$$

- Image f –> Fourier Transform x (and back)

Fourier transform as change of image basis

$$x_{ij} = f \cdot \Phi_{ij} = \sum_{\lambda,\mu} f(\lambda,\mu)\Phi_{ij}(\lambda,\mu)$$

$$f = x_{11}\Phi_{11} + x_{21}\Phi_{21} + x_{12}\Phi_{12} + x_{22}\Phi_{22}$$

$$F(u,v) = \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y)\, e^{-i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)}$$

$$f(x,y) = \frac{1}{MN} \sum_{u=1}^{M} \sum_{v=1}^{N} F(u,v)\, e^{i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)}$$

Compare with ordinary Fourier Transform

Definition

Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. The Fourier transform of f is defined as

$$(\mathcal{F}f)(u) = F(u) = \int_{-\infty}^{+\infty} e^{-i2\pi xu} f(x)dx \ .$$

■

Theorem

Under the right assumptions on f, the following inversion formula

$$f(x) = \int_{-\infty}^{+\infty} e^{i2\pi ux} F(u)du$$

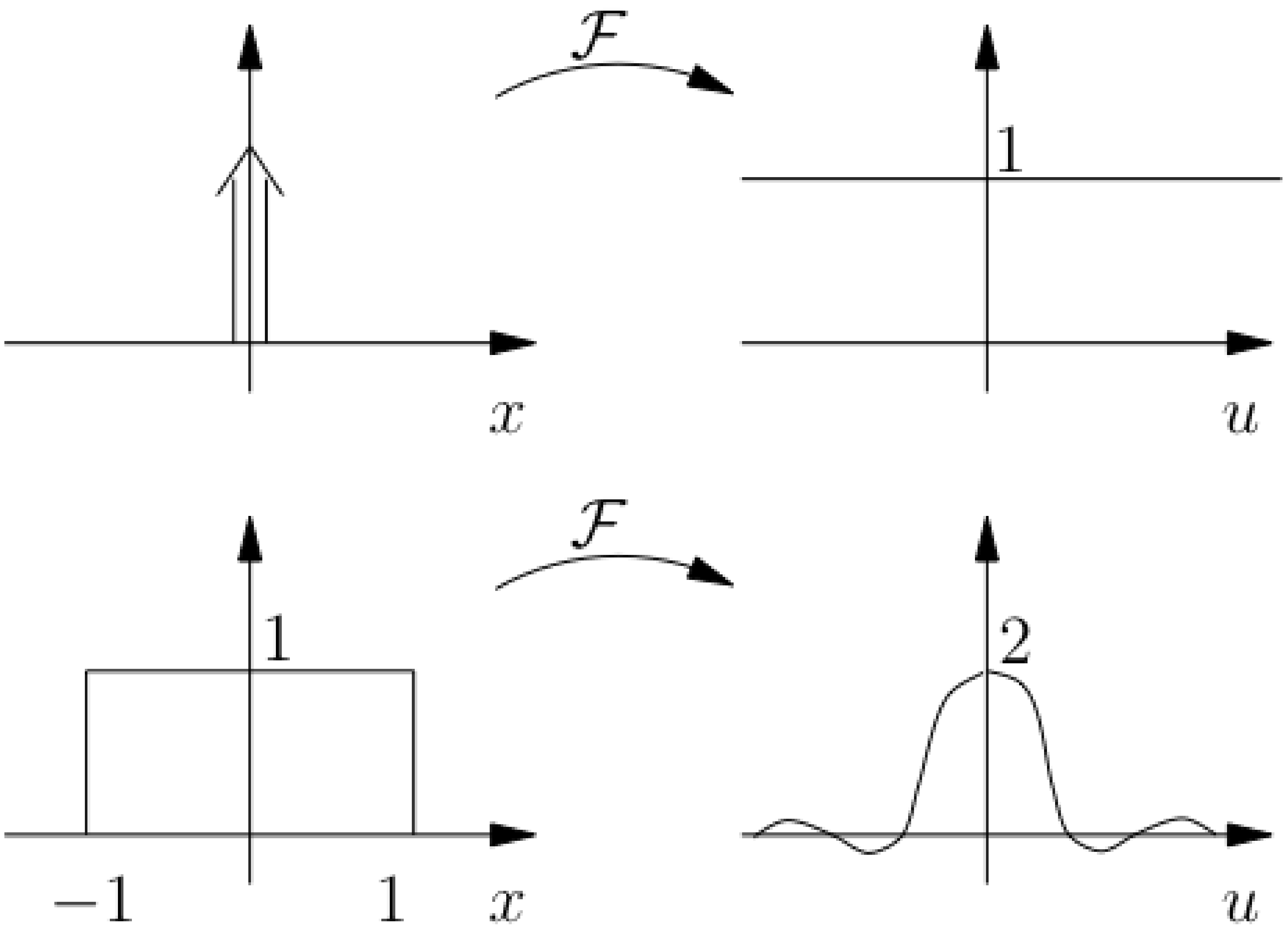
holds.

Examples

$$\delta(x) \mapsto 1(u)$$

$$\mathrm{rect}(x) \mapsto 2\frac{\sin(2\pi u)}{2\pi u} = 2\,\mathrm{sinc}(2\pi u)$$

Examples



Examples

$$
\begin{aligned}
c_1 f_1(x) + c_2 f_2(x) &\mapsto c_1 F_1(u) + c_2 F_2(u) && \text{(linearity)} \\
f(\lambda x) &\mapsto \frac{1}{|\lambda|} F(\frac{u}{\lambda}) && \text{(scaling)} \\
f(x-a) &\mapsto e^{-i2\pi ua} F(u) && \text{(translation)} \\
e^{-i2\pi xa} f(x) &\mapsto F(u+a) && \text{(modulation)} \\
\overline{f(x)} &\mapsto \overline{F(-u)} && \text{(conjugation)} \\
\frac{df}{dx} &\mapsto 2\pi i u F(u) && \text{(differentiation I)} \\
-2\pi i x f(x) &\mapsto \frac{dF}{du} && \text{(differentiation II)}
\end{aligned}
$$

Example: $\delta(x-1) \mapsto e^{-i2\pi u}$

Discrete Fourier Transform - 1D

$$f = \begin{bmatrix} f(1) \\ \vdots \\ f(N) \end{bmatrix}$$

$$F(u) = \sum_{x=1}^{N} f(x)\, exp[-i2\pi(u-1)(x-1)/N]$$

$$F(u) = \sum_{x=1}^{N} f(x)\omega_N^{(x-1)(u-1)}$$

$$\omega_N = \exp(-i2\pi/N)$$

Discrete Fourier Transform - 1D

Definition

The Fourier Matrix $\mathcal{F}_N$ is given by

$$\mathcal{F}_N = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N & \omega_N^2 & \dots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \dots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \dots & \omega_N^{(N-1)(N-1)} \end{pmatrix} .$$

$$f \longrightarrow F = \mathcal{F}_N f.$$

Discrete Fourier Transform - 1D

Theorem 3.3.1. *For the Fourier matrix the following holds,*

$$\mathcal{F}\overline{\mathcal{F}} = NI \ .$$

From this we obtain $\mathcal{F}^{-1} = \frac{1}{N}\overline{\mathcal{F}}$ *The inverse Fourier transform is thus*

$$f = \overline{\mathcal{F}}F \Longleftrightarrow f(x) = \frac{1}{N}\sum_{u=1}^{N} F(u)\omega_N^{(x-1)(u-1)}, \quad x = 1, \ldots, N \ .$$

Discrete Fourier Transform - 1D

- Important: DFT assumes that signals are periodic!
- Think of the signal as wrapped periodically

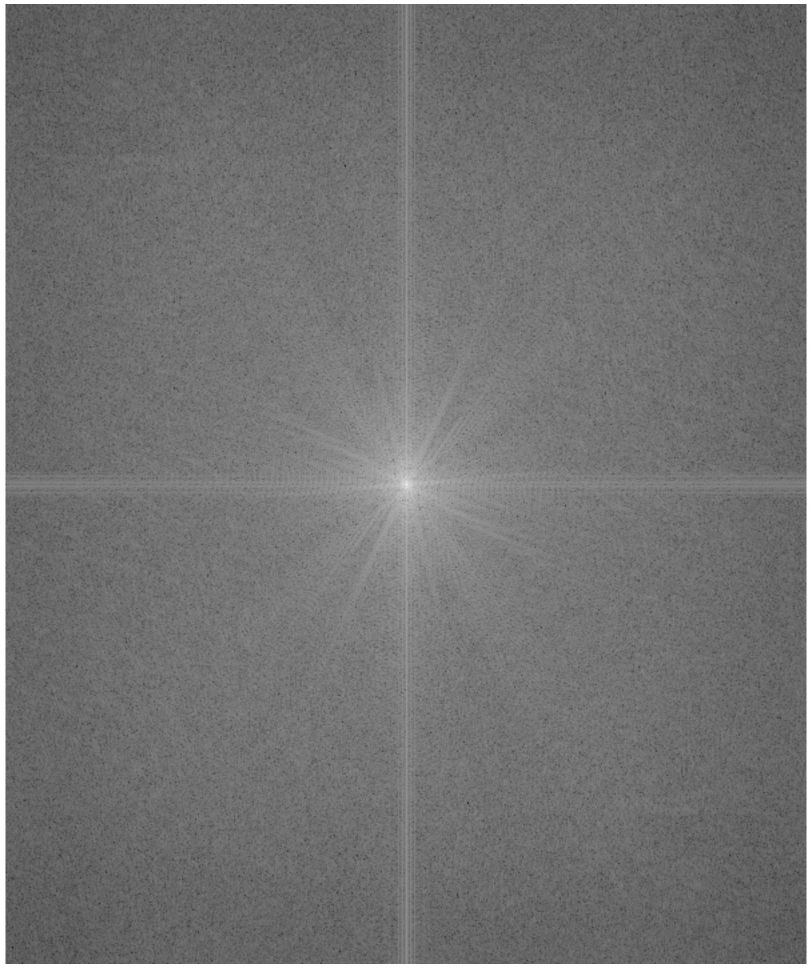
- Fourier transform is complex.
- Plot absolute value and phase

- Low frequencies in the edges/corners.
- Ordinary images typically have large values for low frequencies.

Discrete Fourier Transform - 2D

$$F(u,v) = \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y)\, e^{-i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)}$$

$$f(x,y) = \frac{1}{MN} \sum_{u=1}^{M} \sum_{v=1}^{N} F(u,v)\, e^{i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)}$$



Discrete Fourier Transform - 2D

Let the matrix F represent the Fourier transform of the image $f(x, y)$:

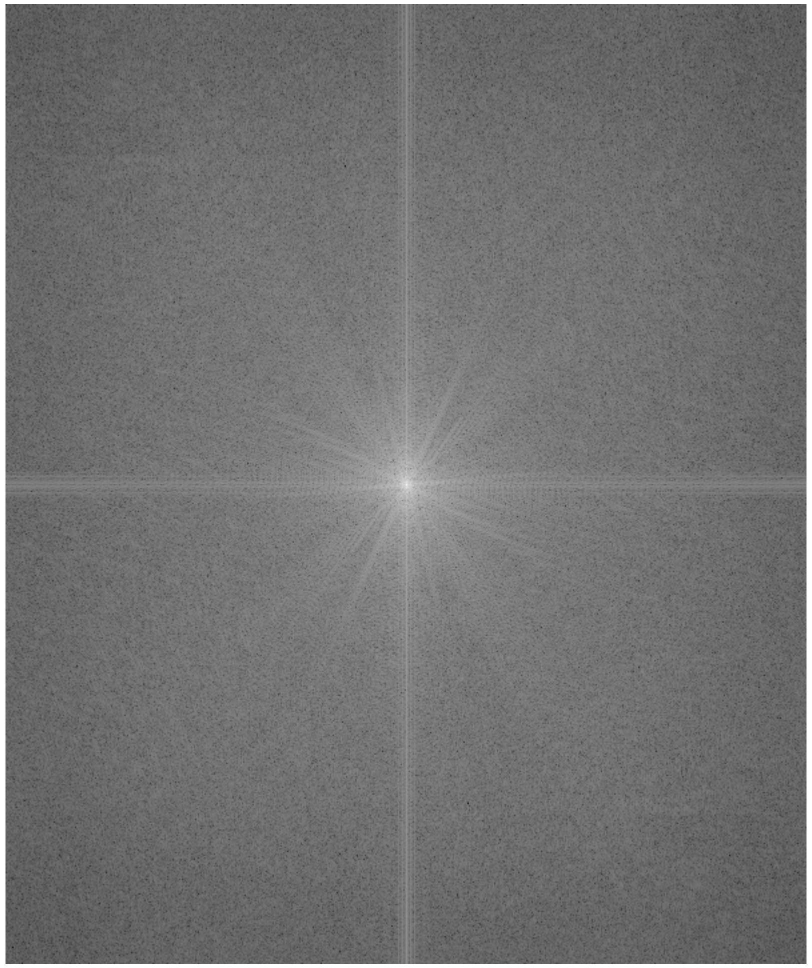
$$F = \mathcal{F}_M f \mathcal{F}_N$$

or

$$F = \mathcal{F}_M (\mathcal{F}_N f^T)^T \quad .$$

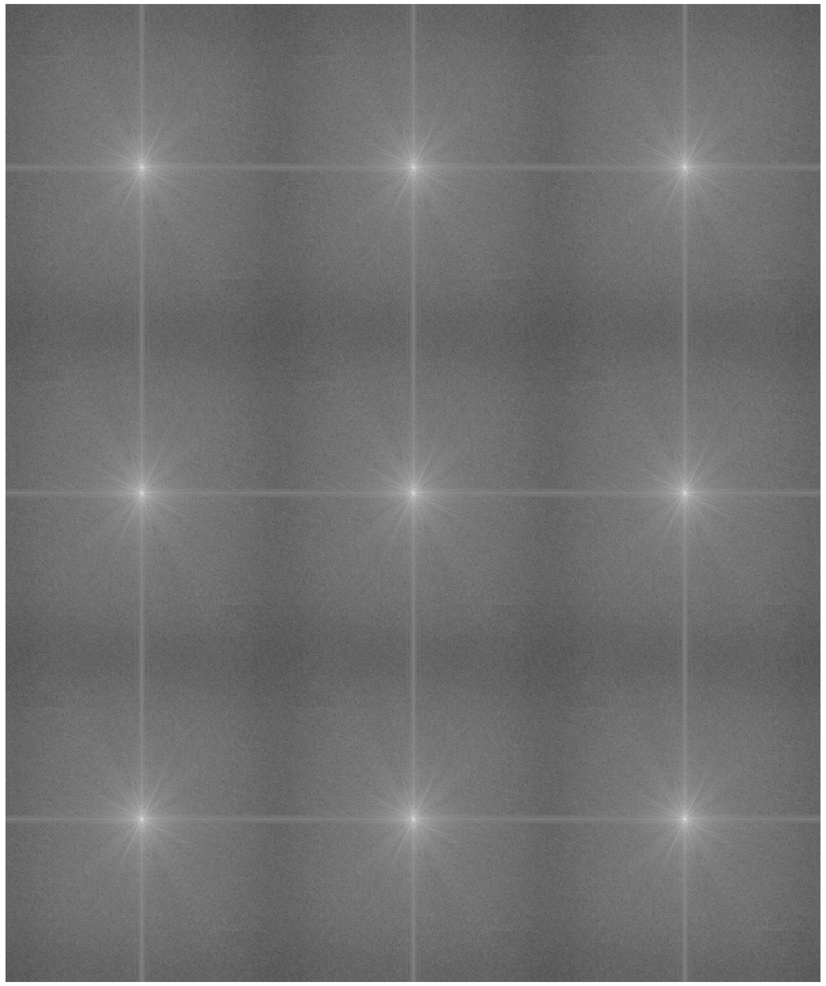
i.e. the DFT in two dimensions can be calculated by repeated use of the one-dimensional DFT, first for the rows, then for the columns.

Discrete Fourier Transform - 2D Example



Fourier transform is complex. Plot absolute value and phase

Discrete Fourier Transform - 2D Example – Periodic expansion



- Usually, the gray-levels of the Fourier Transform images are scaled using $c \log(1 + |F(u, v)|)$.
- The middle of the Fourier image (after fftshift) corresponds to low frequencies.
- Outside the middle high components in F corresponds to higher frequencies and the direction corresponds to "edges"in the images with opposite orientation.

Fourier transform

- Image

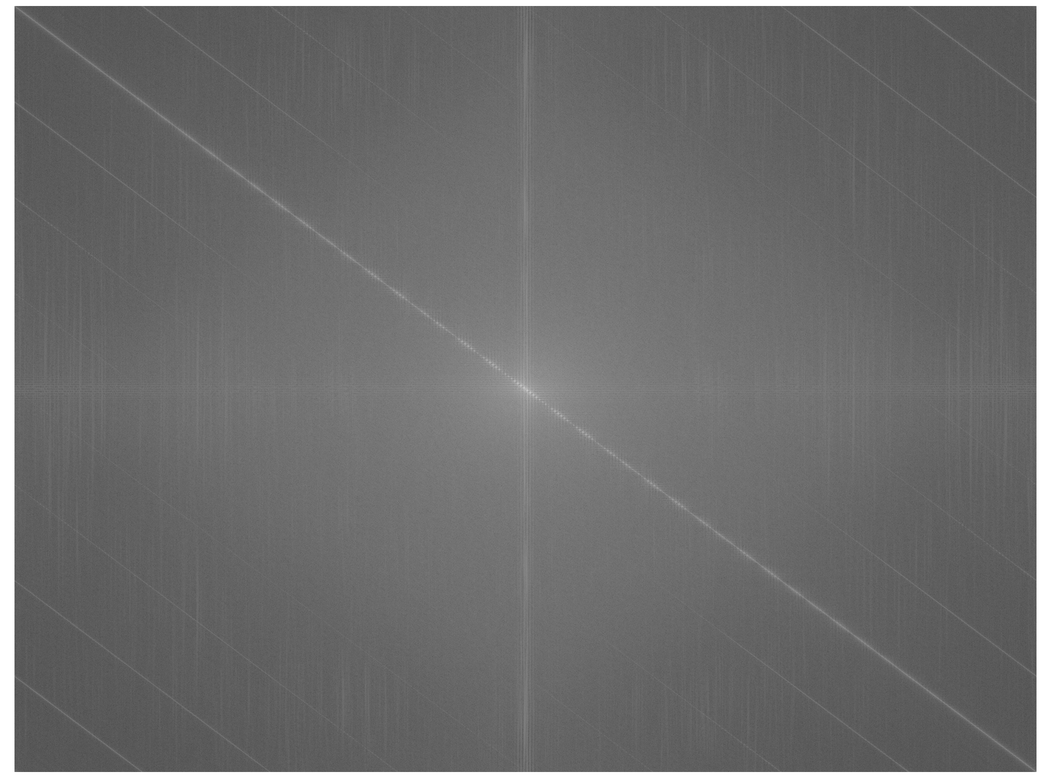
Fourier transform

- Image

- abs(fft2(I))

Fourier transform

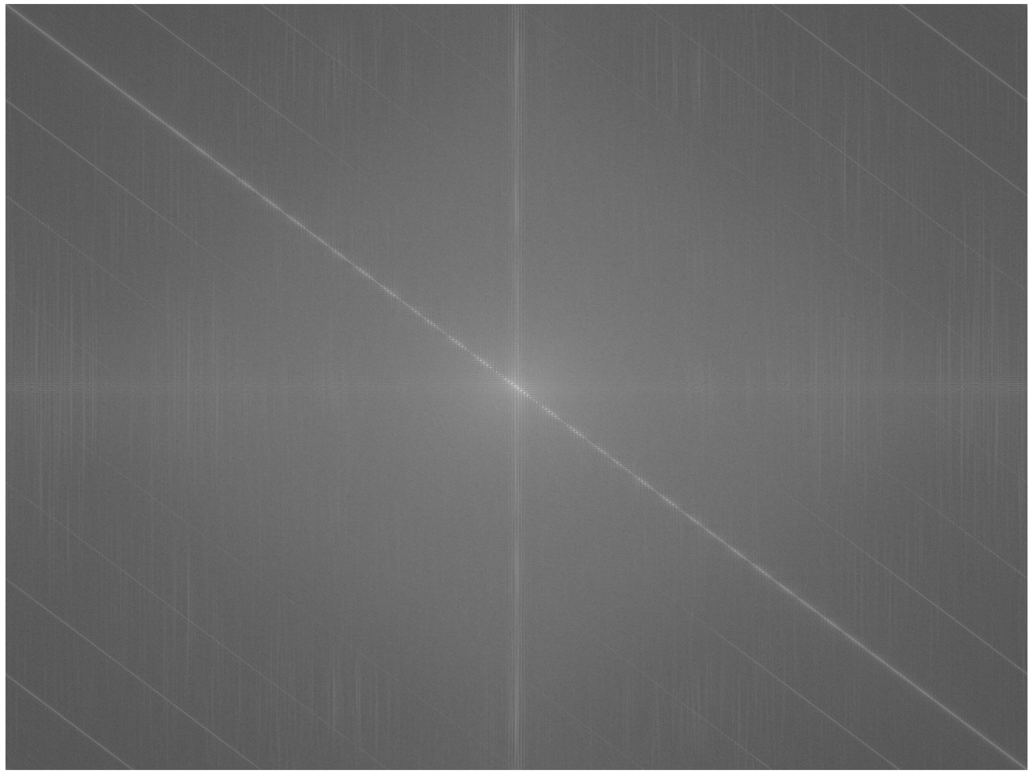
- Image



- log(abs(fft2(I)))

Edge effects

- Image

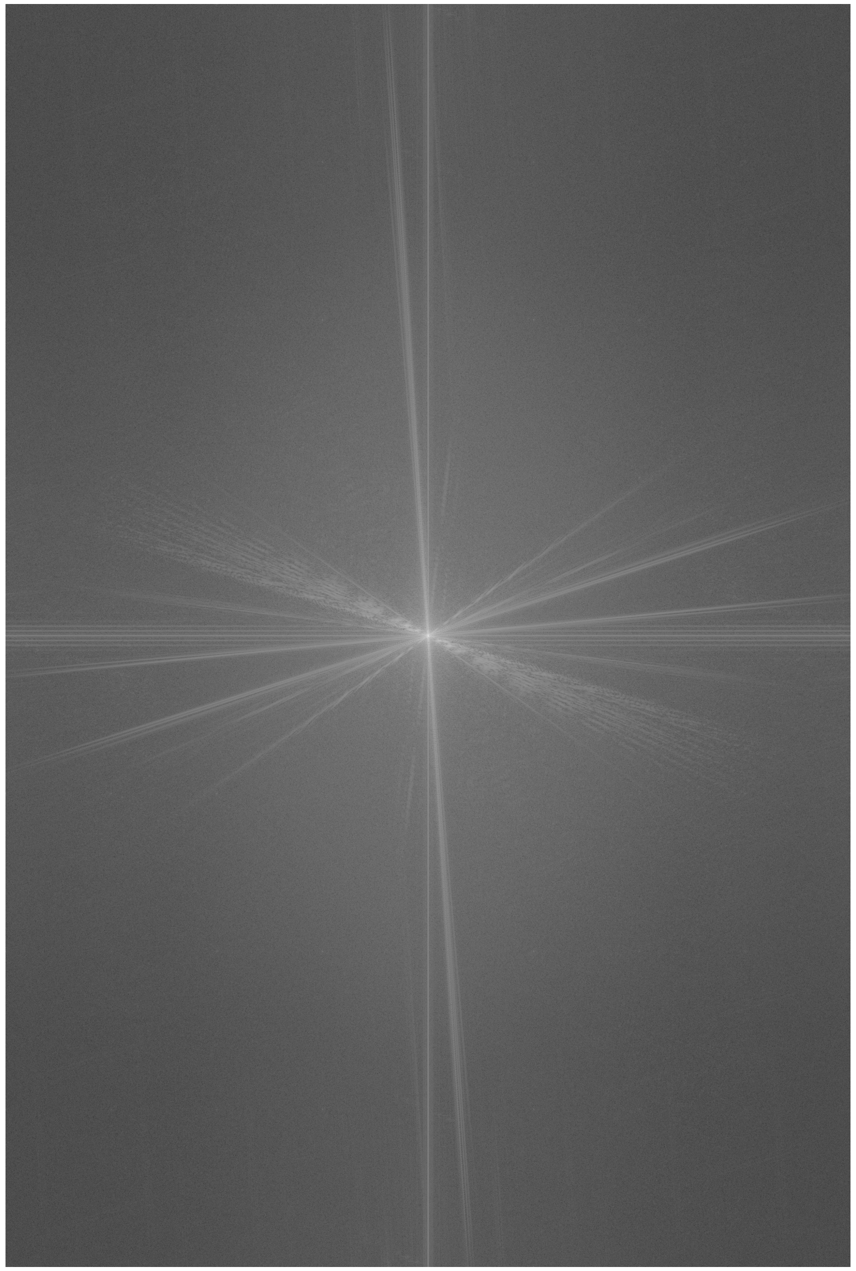


- log(abs(fft2(I)))

Fourier transform

- Image

Fourier transform



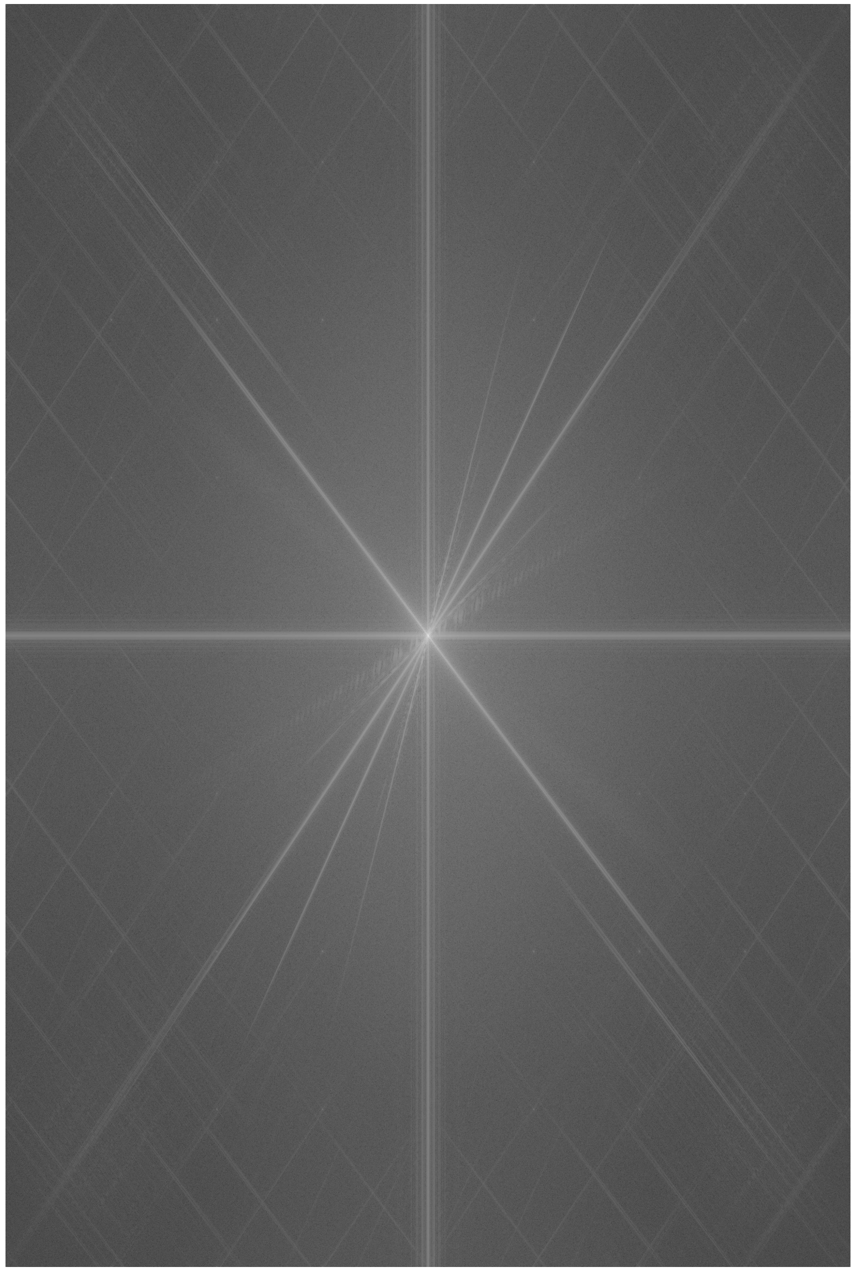
•Image

•Fourier transform

Fourier transform

• Image

Fourier transform



• Image

• Fourier transform

Review

- Linear algebra
  - The space of images is a linear vector space
  - Images are ’vectors’ – in the sense that they are elements of a linear vectors space
  - Can be confusing. Can a matrix be a vector???
- Useful tools
  - Change of basis
  - Projection onto a subspace, onto affine subspace
  - PCA
- Fourier Transform

- Read lecture notes
- Experiment with matlab demo scripts
- Continue with assignment 1

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