LINEAR SYSTEMS — Final Exam

Department of Automatic Control LTH, Sweden.
Instructor: Venkatraman Renganathan
Email: venkat@control.lth.se

Due: $12^{th}$ April, 2022 at 18:00 CET

## Rules for Final Exam

- Please write your name here with date: ____________________
- The final exam will be a take-home 36 hours exam worth 100 points. There will be 5 questions in total. Students should present the entire work leading to their solution. You will be assessed only by your approach to solve the problem.
- The students are allowed to use computer, internet, any books, lecture notes, etc..., but **should strictly do the work individually without collaborating with each other** and the instructor shall be available via email throughout to aide them with any doubts and questions. Plagiarism (copying solutions from others) will never be tolerated at any cost.
- The exam will be due by the evening **18:00 on Tuesday, April 12, 2022**. The deadline for submitting the final exam will be strict and no extensions shall be awarded unless otherwise for unavoidable circumstances.
- Students are strongly encouraged to submit their work in a neatly typed LaTeX document with all supporting code attached to the same document. If typing is not possible, hand-written submissions will be considered as long as they are legibly written with neat presentation. In both the cases, neither multiple submissions nor a submission with multiple files shall be tolerated.
- Students should submit their attempted exam through email to the instructor's email `venkat@control.lth.se` as **one** pdf document and their corresponding submission file should be named as **"FirstnameLastnameFinalExam.pdf"**. For instance, my full name is Venkatraman Renganathan and I would submit my pdf named as **VenkatramanRenganathanFinalExam.pdf**

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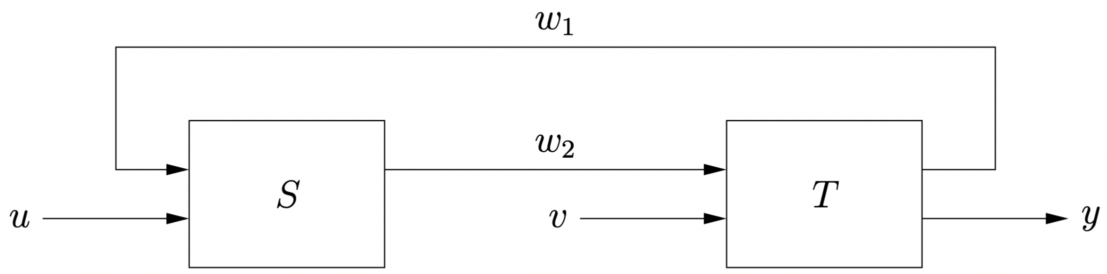
## Grading & Assessment (To be filled by the instructor)

| **Question** | **Score out of 20** |
|---|---|
| 1 | /10 |
| 2 | /20 |
| 3 | /30 |
| 4 | /20 |
| 5 | /20 |
| **Total** | /100 |
| **Weight:** 60% **of Total** | |

| **Student Name:** | |
|---|---|
| **Assessment Criteria** | **Weight in %** |
| Hand-ins | |
| Participation | |
| Final Exam | |
| Total | |
| Course Final Grade | |
| Instructor Signature: | |

## Problem 1: Modelling Interconnected Linear Systems (10 Points)

Consider two linear subsystems $S$ and $T$ interconnected as follows




The subsystems are modelled as follows

$$S : \dot{x} = Ax + B_1 u + B_2 w_1, \quad w_2 = Cx + D_1 u + D_2 w_1 \tag{1}$$

$$T : \dot{z} = Fz + G_1 v + G_2 w_2, \quad w_1 = H_1 z, \quad y = H_2 z + J w_2. \tag{2}$$

Assume that all matrices are of appropriate dimensions. Express the overall system as a single linear dynamical system with input, state, and output given by

$$\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix}, y \tag{3}$$

respectively. Explicitly give the state space model of the overall system.



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## Problem 2: Internal Stability & Gain Margin of LQR (20 points)

1. **$(\epsilon, \delta)$ Condition for Uniform Stability:** Prove that the following CT-LTV linear system

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0 \tag{4}$$

is uniformly stable iff $\forall \epsilon > 0, \exists \delta > 0$, such that

$$\|x_0\| \leq \delta \quad \implies \quad \|x(t)\| \leq \epsilon, \forall t \geq t_0, \tag{5}$$

and the choice of $t_0$ is regardless (arbitrary). It is enough to prove just one direction for full points. If you are smart, you will prove the direction that is easy. If you are enthusiastic, you can prove both directions.

2. **Gain Margin of LQR:** The intent of this problem is to study the robustness of LQR systems. Consider the CT LTI system

$$\dot{x}(t) = Ax + Bu, \tag{6}$$

and a quadratic cost with penalty matrices $Q \succeq 0$ and $R \succ 0$ respectively. Assume that $(A, B)$ is controllable and $(A, Q)$ is observable. Now, consider the closed loop system under full state feedback $u = \alpha Kx$, where $K$ denotes the optimal state feedback gain matrix for the system given by (6) and $\alpha > 0$. Remember that for certain values of $\alpha$, your control input will **not** be LQR optimal and in fact the closed loop system $\dot{x}(t) = (A + \alpha BK)x$ can even be unstable.

   (a) Explain how can you recover the LQR optimal input from the given setting (report the value of $\alpha$).

   (b) For what values of $\alpha$, is the closed loop system stable? Remember that your solution of $\alpha$ is going to be the **gain margin** of LQR systems. **Hint:** Use quadratic Lyapunov function to infer the range of $\alpha$ values.

## Problem 3: Kalman Filter for Gauss-Markov System (30 points)

1. Generate a **random** Gauss-Markov System with $n = 10$. That is,

$$x_{k+1} = Ax_k + w_k, \tag{7}$$

where $A \in \mathbb{R}^{n\times n}$ should be random and Schur stable. Scale $A$ matrix so that its spectral radius is 0.95. The noise $w_k \in \mathbb{R}^{10}$ should be IID and draw the process noise from $w_k \sim \mathcal{N}(0, \Sigma_w)$. For the above $A$ matrix, generate a random $\Sigma_w \succeq 0$ such that the pair $(A, W)$ is controllable and use it. The initial state is uncertain as well and draw them as $x_0 \sim \mathcal{N}(0, \Sigma_{x_0})$. Generate a random $\Sigma_{x_0} \succ 0$ and use it. After fixing the covariances, create $N = 50$ state trajectories of each starting from $x_0 \sim \mathcal{N}(0, \Sigma_{x_0})$ and plot only $(x_k)_1$ for $T = 100$ time steps (first component of state at all 100 time steps) for all the 50 trajectories. Denote the mean and covariance of state $x_k$ at any time instant $k$ as $\mathbb{E}[x_k] = \bar{x}_k$ and $\mathbb{E}[(x_k - \bar{x}_k)(x_k - \bar{x}_k)^\top] = \Sigma_{x_k}$ respectively. Solve and report the asymptotic state covariance $\Sigma_x$. Submit all your code and plots with discussion.

2. Now additionally generate a random output matrix $C \in \mathbb{R}^{p\times n}$ with $p = 3$ and ensure that $(A, C)$ is observable. That is, with the same $A$ matrix obtained above, obtain a $C$ matrix such that $(A, C)$ is observable. This leads us to an output equation

$$y_k = Cx_k + v_k, \tag{8}$$

with $v_k \in \mathbb{R}^p$ and draw the sensor noise from $v_k \sim \mathcal{N}(0, \Sigma_v)$. Generate a random $\Sigma_v \succ 0$ and use it. Design a Kalman filter for the above system and simulate the system with the filter for $T = 100$ time steps. Plot the following quantities for all $k = 1, 2, \ldots, T$ (all in the same figure) $\mathbb{E}\left[\|x_k\|^2\right]$, $\mathbb{E}\left[\|x_k - \hat{x}_k\|^2\right]$, $\sqrt{\mathbb{E}\left[\|x_k\|^2\right]}$, $\sqrt{\mathbb{E}\left[\|x_k - \hat{x}_k\|^2\right]}$. Submit all your code and plots with discussion.

3. Let us call the above Gauss-Markov system and its output equation as **nominal system**. Let $L$ denote the steady-state Kalman filter gain for the nominal system. Recall that $L$ exists as nominal system is both controllable and observable. Let us denote the steady state error covariance matrix of the nominal system as $\hat{\Sigma}$ where $\hat{\Sigma}$ is obtained using the Kalman filter that you designed for the nominal system. Now consider the **perturbed system**

$$x_{k+1} = (A + \Delta A)x_k + w_k, \quad y_k = Cx_k + v_k, \tag{9}$$

where $\Delta A \in \mathbb{R}^{n\times n}$ is a random perturbation of the system matrix. For simplicity, generate a random but sparse $\Delta A$ with exactly just $n$ non-zero elements and use it for simulation. The intention of this part is to examine what happens when you design a Kalman filter for the nominal system (which you have already done in part 2), and use it for the perturbed system (9).

   (a) Find steady state values of $\mathbb{E}\left[\|x_k\|^2\right]$ both for nominal & the perturbed system.

   (b) Find $\sqrt{\mathbb{E}\left[\|x_k - \hat{x}_k\|^2\right]}$, where $x_k$ is the state of the perturbed system, and $\hat{x}_k$ is the state estimate of the nominal system from the Kalman filter. Compare this to $\mathtt{Tr}[\hat{\Sigma}]$ which gives the steady state value of $\sqrt{\mathbb{E}\left[\|x_k - \hat{x}_k\|^2\right]}$ when $x_k$ is the state of the nominal system.

## Problem 4: Riccati Equation ⇔ Lyapunov Equation (20 points)

1. **Matrix Identities Using Sherman-Morrison Woodbury Matrix Inversion Lemma**

$$(A + U\Sigma V^\top)^{-1} = A^{-1} - A^{-1}U(\Sigma^{-1} + V^\top A^{-1}U)^{-1}V^\top A^{-1} \tag{10}$$

A simple and reduced matrix inversion lemma is given by taking $A = \Sigma = I$.

$$(I + UV^\top)^{-1} = I - U(I + V^\top U)^{-1}V^\top \tag{11}$$

Prove the following

(a) `Inverse of a Sum of Two Matrices:`

$$(I + A)^{-1} = I - (I + A)^{-1}A = I - A(I + A)^{-1} \tag{12}$$

(b) `Inverse of a Sum of matrix inverse and another matrix:`

$$(A^{-1} + B)^{-1} = A(I + BA)^{-1} = (I + AB)^{-1}A \tag{13}$$

(c) `Difference of Matrix Inverses:` Assume $A, B \in \mathbb{R}^{n\times n}$ to be invertible

$$A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1} \tag{14}$$

(d) `Push-Pop Identity:`

$$P(I + KP)^{-1} = (I + PK)^{-1}P \tag{15}$$

(e) `Riccati Aide:`

$$G - G(P^{-1} + G)^{-1}G = (P + G^{-1})^{-1} \tag{16}$$

2. **Equivalence between Riccati & Lyapunov Equations**
The solution to the infinite horizon DT LQ optimal control problem with quadratic cost $J = \sum_{k=0}^{\infty}\left(x_k^\top Q x_k + u_k^\top R u_k\right)$ and linear dynamics $x_{k+1} = Ax_k + Bu_k$ is given by $u_k = Kx_k, \forall k \in \mathbb{N}$ where

$$K = -(R + B^\top PB)^{-1}B^\top PA \tag{17}$$

and $P$ solves the following discrete-time algebraic Riccati equation (DARE)

$$P = Q + A^\top PA - A^\top PB(R + B^\top PB)^{-1}B^\top PA. \tag{18}$$

Prove the following:

(a) Show that with $G = BR^{-1}B^\top$ and above matrix identities that the DARE given by (18) can be expressed equivalently as the following Lyapunov equations

$$P = Q + A^\top P(I + GP)^{-1}A, \tag{19}$$

$$P = Q + A^\top (P^{-1} + G)^{-1}A. \tag{20}$$

(b) Show that with $Y = P^{-1}, W = Q^{-1}, V = R^{-1}$, the above DARE equation can be expressed equivalently in the **"inverse form"** as

$$Y = W - WA^\top(Y + AWA^\top + BVB^\top)^{-1}AW. \tag{21}$$

## Problem 5: Observability & Detectability (20 Points)

Consider the following CT LTI system

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} x(t), \tag{22}$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t). \tag{23}$$

1. Infer the observability of the above system using all four tests of observability.
2. Find the unobservable subspace $\mathfrak{UO}[t_0, t_f]$.
3. Check and report if the system is detectable.
4. Design a Luenberger observer to place the eigenvalues of the closed loop estimation error dynamics at $-1, -1$ and find $\lim_{t \to \infty} e(t)$.
5. Suppose that the $A$ matrix is changed to $\bar{A} = \begin{bmatrix} -0.9 & 1 \\ 0 & 1 \end{bmatrix}$. Will the observer designed in the previous part yield a converging and an asymptotically stable estimation error dynamics with this $\bar{A}$ matrix?