Intent of this exercise

The motive of this exercise session is try out some problems in realization theory & popular operators in control theory to get a good grasp of that content. This specific exercise session will aide the students to successfully complete the Hand-in # 5,6. The following exercise problems are suggested for students to be solved.

Problem 1
Solve the following optimal control problem and find the optimal control,

$$\begin{aligned} \min \quad & x(1)^2 + \int_0^1 u(t)^2 dt, \\ \text{s.t.} \quad & \dot{x}(t) = e^{-t}u(t), \\ & x(0) = 1. \end{aligned} \tag{1}$$

Hint: Use a differential Riccati equation to solve.

Problem 2
Let us investigate the relative degree and zeros of single-input-single- output systems (SISO). Consider the state space representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

and its corresponding transfer function with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1}, C \in \mathbb{R}^{1\times n}$.

$$G(s) = C(sI - A)^{-1}B = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

1. The relative degree $d = n - m$ is the excess degree of the denominator compared to the numerator. Prove that

$$CA^kB = 0, \quad \forall k = 0, 1, \ldots, d-2 \quad \text{and} \quad CA^{d-1}B \neq 0.$$

2. Factorise the transfer function using the poles and zeros as

$$G(s) = C(sI - A)^{-1}B = b_m \frac{(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)}$$

Let the stationary state space solution has the form $x(t) = \bar{x}e^{st}$. Suppose that a complex exponential input $u(t) = \bar{u}e^{st}, \bar{u} \neq 0$ gives rise to complex exponential output $y(t) = G(s)\bar{u}e^{st}$. If the frequency is equal to the zeros

$s = z_k, k = \{1, \ldots, m\}$, then the output goes to zeros. Prove that the zeros can be characterized as those complex numbers s such that

$$\texttt{rank}\left(\begin{bmatrix} -sI + A & B \\ C & 0 \end{bmatrix}\right) < n + 1.$$

3. Use your results from above to verify that the transfer function $G(s) = C(sI - A)^{-1}B$ has the relative degree $d = 1$ and a zero at $z = 2$ for the following system.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} -2 & 1 \end{bmatrix}.$$

Problem 3

1. Given a minimal realisation (A, B, C), show that $(A + BLC, B, C)$ is also minimal with any L.
2. Check whether the following system is minimal or not and discuss when it is BIBO stable.

$$A = \begin{bmatrix} -(1+a) & -a \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

3. Given a minimal realisation (A, B, C), is $(A + BK, B, C)$ also minimal with any K?

Problem 4

Compute the Smith-McMillan form and the McMillan degree of the transfer function matrix

$$H(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{s}{s+1} \\ \frac{1}{s+3} & \frac{s}{s+3} \end{bmatrix}.$$