5. Realization Theory, MIMO Poles & Zeros

- Minimal Realisation
  - Markov Parameters
  - Similarity of Minimal Realisations
  - Balanced Reduction
- Stable Factorisation of Polynomials
  - Left & Right Co-prime Factorisation
- MIMO Poles & Zeros
  - Transmission Zeros

Realisation Theory

Realisation of LTI System

Given a transfer function $G(s)$, the CT LTI state space system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

is a **realisation** of $G(s)$ if $G(s) = C(sI - A)^{-1}B + D$.

- Denote it by $G \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$
- Size n of state vector x is called the **order** of realisation
- $G(s)$ can have multiple realisations and of different orders too!

Question

Given transfer function $G(s)$, what is the smallest order of its realisation?

Minimal Realisation

Minimal Realisation

Given transfer function $G(s)$, a realisation is called the **minimal** if there is no other realisation of $G(s)$ that is of smaller order.

- The transfer function corresponding to minimal realisation will not have any pole-zero cancellation. If there is one - then it's not minimal.
- Minimal realisation closely related to controllability & observability.

Theorem of Minimal Realisation for LTI System

Every minimal realisation must be both controllable & observable.

Proof: If a realisation is not controllable or not observable, we can use the Kalman decomposition to obtain a realisation of smaller order.

Markov Parameters & Impulse Response

$$\mathcal{L}\left[\frac{t^i}{i!}\right] = s^{-(i+1)} \quad \Longrightarrow \; (sI - A)^{-1} = \mathcal{L}\left[\sum_{i=0}^{\infty} \frac{t^i}{i!} A^i\right] = \sum_{i=0}^{\infty} s^{-(i+1)} A^i$$

$$\Longrightarrow \; G(s) = C(sI - A)^{-1}B + D = \sum_{i=0}^{\infty} s^{-(i+1)} CA^iB + D$$

Markov Parameters

- The matrices $D, CA^iB, i \geq 0$ are called the **Markov Parameters**
- System's impulse response is given by

$$G(t) = \mathcal{L}^{-1}\left[G(s)\right] = \mathcal{L}^{-1}\left[C(sI - A)^{-1}B + D\right] = Ce^{At}B + D\delta(t).$$

Further, $\frac{d^iG(t)}{dt^i} = CA^ie^{At}B, \quad \forall i \geq 1, t > 0 \implies \lim_{t \downarrow 0^+} \frac{d^iG(t)}{dt^i} = CA^iB.$

Zero-State Equivalence of Two Realisations

Theorem

Two realisations

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du. \end{cases} \qquad \begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ \bar{y} = \bar{C}\bar{x} + \bar{D}u. \end{cases} \tag{1}$$

are zero-state equivalent **iff** they have the same Markov parameters,

$$D = \bar{D}, \quad CA^iB = \bar{C}\bar{A}^i\bar{B}, \quad \forall i \geq 0.$$

Proof: If Markov parameters are same, then both realisations realise the same transfer function. Conversely, if both realisations realise the same transfer function, then they must have same $D = \lim_{s\to\infty} G(s)$ & they must have same impulse response $\implies$ their Markov parameters are same.

Proof Other direction for Theorem of Minimal Realisation

Proof by contradiction

- Assume that (A, B, C, D) with order n is both controllable & observable but not minimal. Then, by Kalman Decomposition theorem, $\exists(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ with order $\bar{n} < n$ is minimal.
- Since $\mathcal{C}, \mathcal{O}$ are of full rank and so is $\mathcal{O}\mathcal{C}$.
- Similarly form $\bar{\mathcal{O}}\bar{\mathcal{C}}$ and observe that $\texttt{rank}(\bar{\mathcal{O}}\bar{\mathcal{C}}) = \bar{n} < n$
- However, both (A, B, C, D) and $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ realise the same transfer function with same Markov parameters. Hence, $\mathcal{O}\mathcal{C} = \bar{\mathcal{O}}\bar{\mathcal{C}}$.
- But $\texttt{rank}(\mathcal{O}\mathcal{C}) = n$ and $\texttt{rank}(\bar{\mathcal{O}}\bar{\mathcal{C}}) = \bar{n} < n$. CONTRADICTION

Similar Minimal Realisations

Theorem

All minimal realisations of a transfer function are similar to each other.

Proof:

- Suppose that $(A, B, C, D), (\bar{A}, \bar{B}, \bar{C}, \bar{D})$ are two minimal realisation of $G(s)$. Then, Markov parameters & Hankel Matrices are equal.

$$D = \bar{D}, \quad CA^iB = \bar{C}\bar{A}^i\bar{B}, \quad \forall i \geq 0, \text{ and } \mathcal{O}\mathcal{C} = \bar{\mathcal{O}}\bar{\mathcal{C}}.$$

Define using the pseudo-inverses the required transformation

$$T = \mathcal{C}\bar{\mathcal{C}}^\dagger = \mathcal{O}^\dagger\bar{\mathcal{O}} \implies T^{-1} = \bar{\mathcal{C}}\mathcal{C}^\dagger = \bar{\mathcal{O}}^\dagger\mathcal{O}$$

$$\implies AT = T\bar{A}, B = T\bar{B}, \bar{C} = CT$$

$\implies (A, B, C, D), (\bar{A}, \bar{B}, \bar{C}, \bar{D})$ are similar to each other.

Balanced Realization

Popular approach for obtaining a minimal realisation where a new state-space description is got using the Hankel singular values so that the Controllability & Observability Gramians are diagonalised.

- Consider a system $G(s)$ with minimal realisation $G \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$.
- Infer the Gramians $W_{\mathfrak{C}}, W_{\mathfrak{O}}$ by solving the Lyapunov equations

$$AW_{\mathfrak{C}} + W_{\mathfrak{C}}A^\top + BB^\top = 0$$

$$W_{\mathfrak{O}}A + A^\top W_{\mathfrak{O}} + C^\top C = 0$$

Hankel Singular Values

Hankel Singular Values measures energy for each state in a system

$$\sigma_i := \sqrt{\lambda_i\left(W_{\mathfrak{C}}W_{\mathfrak{O}}\right)}, \quad \forall i = 1, \ldots, n.$$

Balanced Realization

Find a transformation T that results in controllability and observability gramians being equal & diagonal with the diagonal entries equal to the **Hankel** singular values. That is, look for T such that

$$G \sim \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right] \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \quad \text{and}$$

$$\overline{W}_{\mathfrak{C}} = \overline{W}_{\mathfrak{O}} = \Sigma = \mathtt{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n), \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n.$$

Look at the Lyapunov equations for transformed system.

$$\overline{A}\,\overline{W}_{\mathfrak{C}} + \overline{W}_{\mathfrak{C}}\overline{A}^{\top} + \overline{B}\overline{B}^{\top} = 0, \quad \overline{W}_{\mathfrak{O}}\overline{A} + \overline{A}^{\top}\overline{W}_{\mathfrak{O}} + \overline{C}^{\top}\overline{C} = 0, \text{ where,}$$

$$\overline{W}_{\mathfrak{C}} = T^{-1}W_{\mathfrak{C}}T^{-\top}, \quad \text{and} \quad \overline{W}_{\mathfrak{O}} = T^{\top}W_{\mathfrak{O}}T$$

$$\Longrightarrow \overline{W}_{\mathfrak{C}}\overline{W}_{\mathfrak{O}} = \left(T^{-1}W_{\mathfrak{C}}T^{-\top}\right)\left(T^{\top}W_{\mathfrak{O}}T\right) = T^{-1}W_{\mathfrak{C}}W_{\mathfrak{O}}T = \Sigma^2$$

Balanced Realization

How can we check that $\overline{W}_{\mathfrak{C}} = \overline{W}_{\mathfrak{O}} = \Sigma$? Factorise $W_{\mathfrak{O}} = R^\top R, \exists R^{-1}$.

$$T^{-1} W_{\mathfrak{C}} R^\top R T = \Sigma^2 \quad \Longleftrightarrow \quad (RT)^{-1} R W_{\mathfrak{C}} R^\top (RT) = \Sigma^2.$$

$RW_{\mathfrak{C}}R^\top \succ 0$ is similar to Σ^2. Hence, $\exists$ orthogonal transformation U such that $U^\top U = I$ and $RW_{\mathfrak{C}}R^\top = U\Sigma^2 U^\top$. Then, define the transformation T by setting $(RT)^{-1} U \Sigma^{\frac{1}{2}} = I$. Then,

$$T = R^{-1} U \Sigma^{\frac{1}{2}} \iff T^{-1} = \Sigma^{-\frac{1}{2}} U^\top R, \implies T^{-1} T = I.$$

$$\begin{aligned}
\implies \overline{W}_{\mathfrak{C}} &= T^{-1} W_{\mathfrak{C}} T^{-\top} = (\Sigma^{-\frac{1}{2}} U^\top R) W_{\mathfrak{C}} (R^\top U \Sigma^{-\frac{1}{2}}) \\
&= (\Sigma^{-\frac{1}{2}} U^\top) U \Sigma^2 U^\top (U \Sigma^{-\frac{1}{2}}) = \Sigma \\
\implies \overline{W}_{\mathfrak{O}} &= T^\top W_{\mathfrak{O}} T = (R^{-1} U \Sigma^{\frac{1}{2}})^\top W_{\mathfrak{O}} (R^{-1} U \Sigma^{\frac{1}{2}}) \\
&= (\Sigma^{\frac{1}{2}} U^\top)(R^{-\top} R^\top R R^{-1})(U \Sigma^{\frac{1}{2}}) = \Sigma
\end{aligned}$$

Model Reduction

Goal of model-reduction: Obtain a model G_r from G such that

$$\|G - G_r\|_\infty = \sup_{u \in L_2} \frac{\|(G - G_r)u\|}{\|u\|} \quad \text{is small.}$$

- Given an asymptotically stable system $G \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$.
- Get a balanced realisation $G \sim \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$ via transformation T with $\overline{W}_{\mathfrak{C}} = \overline{W}_{\mathfrak{O}} = \Sigma$.
- Then, in the new coordinates $\bar{x} = \begin{bmatrix} \bar{x}_1 & \dots & \bar{x}_n \end{bmatrix}^\top$, $\bar{x}_1(\bar{x}_n)$ denotes the most (least) controllable & observable coordinate.
- For model reduction, truncate the system when Hankel singular values fall below certain threshold so that $\|G - G_r\|_\infty$ is small.

Balanced Model Reduction

Partition Σ as $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$, where Σ_2 contains smaller Hankel singular values. Then, we can partition G as $G \sim \left[\begin{array}{cc|c} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 \\ \bar{A}_{21} & \bar{A}_{22} & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{C}_2 & D \end{array}\right]$.

The following Lyapunov equations hold for the transformed system

$$\bar{A}\Sigma + \Sigma\bar{A}^\top + \bar{B}\bar{B}^\top = 0 \quad \text{and} \quad \bar{A}^\top\Sigma + \Sigma\bar{A} + \bar{C}^\top\bar{C} = 0.$$

Expanding using above partitions, we can extract two subsystems

$$G_1 \sim \left[\begin{array}{c|c} \bar{A}_{11} & \bar{B}_1 \\ \hline \bar{C}_1 & D \end{array}\right], \quad G_2 \sim \left[\begin{array}{c|c} \bar{A}_{22} & \bar{B}_2 \\ \hline \bar{C}_2 & D \end{array}\right]$$

Balanced Model Reduction

- Usually, when systems have $n = 10^6$ states, only a small number, say $n_r = 50 << n$ will have their Hankel singular values significant.
- Better to approximate system with $\mathbb{R}^{10^6}$ using $\mathbb{R}^{50}$ reduced system.

Theorem

Suppose $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is a balanced realization and

$$\bar{A}_r = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{22}, \quad \bar{B}_r = \bar{B}_1 - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2$$

$$\bar{C}_r = \bar{C}_1 - \bar{C}_2\bar{A}_{22}^{-1}\bar{A}_{21}, \quad \bar{D}_r = D - \bar{C}_2\bar{A}_{22}^{-1}\bar{B}_2$$

Then $\bar{A}_r$ is Hurwitz & $(\bar{A}_r, \bar{B}_r, \bar{C}_r, \bar{D}_r)$ is a minimal & balanced realisation of G_r with Gramian Σ_1 and

$$\sigma_{r+1} \leq \|G - G_r\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i.$$

Lecture - 5

Minimal Realization

Markov Parameters

Similarity of Minimal Realizations

✓ MIMO Poles & Zeros

Transmission Zeros (More on this later)

✓ Stable Factorization of Polynomials

Left & Right Co-prime

Stable Factorization

Consider a transfer function $G(s) = \dfrac{n(s)}{d(s)}$, where $n(s)$, $d(s)$ are numerator, denominator polynomial in s.

Let us study factorisation of such polynomials.

Greatest Common Divisor (GCD)

Let $d \in \mathbb{Z}$, $a, b \in \mathbb{N}$. Then

$d = \gcd(a,b)$ if

i) $d \mid a$, $d \mid b$ (d divides a, b)

ii) If $s \mid a$, $s \mid b$, then $s \mid d$.

Eg:- $a = 18, b = 24$. $d = 6$, $s = 2$ or 3.

iii) If $d = \gcd(a,b)$, then so is $-d$.

iv) If $d = \gcd(a,b)$, then set of all gcd $= \{ud : u \in \mathcal{U}\}$.

Unit

$u \in \mathbb{Z}$ is a unit if $\exists\, v \in \mathbb{Z}$ such that $uv = 1$.

Let $\mathcal{U}$ = set of units $= \{1, -1\}$

Euclidean Division

If $p, q \in \mathbb{Z}$ with $|p| > |q|$, then $\exists\, s \in \mathbb{Z}$ s.t. $|p - sq| < |q|$

Theorem (Bézout's Identity)

If $d = \gcd(a,b)$, then $\exists\, x, y \in \mathbb{Z}$ s.t. $\boxed{xa + yb = d}$

Eg:- $1 = \gcd(3,4)$, then $3x + 4y = 1 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \ldots \infty$ many

Claim: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + r\begin{pmatrix} b \\ -a \end{pmatrix}$

We know $ax + by = 1$ if a, b are co-prime.

$\Leftrightarrow ax = 1 - by = 1 \bmod b$

Eg:- $(5,3)$ are co-prime and $\begin{array}{r} 1 \\ 3\overline{)5} \\ 3 \\ \hline 2 \end{array}$ $(-)$ $\Rightarrow \bmod(5,3) = 2$.

Stable Rational Functions (SRF)

Open Left Half Plane (or) Open (LHP) $= \{ s < 0 \}$

Closed Right Half Plane (or) Closed (RHP) $= \{ s \geq 0 \}$

$S :=$ set of stable rational functions (proper and poles in open LHP).

$S :=$ ring i.e. if $f, g \in S$, so are $f+g$ and $f \cdot g$.

Unit: $U := \{ f \in S : \exists g \in S$ with $f(s) \cdot g(s) = 1, \forall s \}$

Lemma: $f \in U$ iff

a) relative degree $(f) = 0$

b) all zeros of f lie in open LHP.

GCD of (SRF): Let $f, g, h \in S$. $h = \gcd(f, g)$ if

i) $h | f$, $h | g$ i.e. f/h, g/h are still $\in S$.

ii) If $q | f$, $q | g$, then $q | h$

Degree of SRF : Let $f \in S$.

$\delta(f) :=$ relative degree $\}$ + # of closed RHP zeros.

Co-prime:

$n(s), d(s) \in S$ are co-prime if the only common divisors are units.

Theorem: $n(s), d(s) \in S$ are co-prime iff they have NO common zeros at infinity or in the closed RHP.

$\Leftrightarrow \exists x, y \in S$ s.t $\boxed{xf + yg = 1}$, $\forall s$.

Theorem: $h(s)$ is a unit, $h \in U \Leftrightarrow \delta(h(s)) = 0$

Theorem: $h, f \in S$. $h | f$ iff

a) Every closed RHP zero of h is also a closed RHP zero of f of atleast the same multiplicity.

b) Rel. degree of $(h) \leq$ Rel. degree of (f).

If $h | f$, then $\delta(h) \leq \delta(f)$.

Theorem: Suppose $f, g \in S$. Then f, g are coprime if $h | f$, $h | g \Rightarrow h \in U$. i.e. only tranfer fn. that divides f & g must be unit

Examples

	f	$\delta(f)$
1.	$\frac{s-1}{s+1}$	1
2.	$\frac{s-1}{(s+1)(s+2)}$	2
3.	$\frac{s+1}{s+2}$	0
4.	$\frac{s}{(s+3)(s+6)}$	2

f	g	co-prime ?
$\frac{(s-1)}{(s+2)(s+3)}$	$\frac{(s-2)}{(s+4)}$	YES
	$\frac{(s-2)}{(s+4)(s+7)}$	NO bcoz common zero@∞.
1	3	YES
1	2	NO
2	4	NO.

h	f	$h \mid f$?
$\frac{(s-1)(s+3)}{(s+2)(s+7)}$	$\frac{(s-1)(s-2)}{(s+4)(s+5)}$	YES
$\frac{(s-1)}{(s+2)(s+7)}$	$\frac{(s-1)(s-2)}{(s+4)(s+5)}$	?
$\frac{(s-1)^2}{(s+2)(s+7)}$	$\frac{(s-1)(s-2)}{(s+4)(s+5)}$	?

Definition: Suppose $\phi(s)$ is rational. Then (n, d) is a co-prime factorisation of ϕ if

a) $\phi(s) = \dfrac{n(s)}{d(s)}, \ \forall s,$

b) n & d are co-prime

c) $\exists x, y$ s.t $xn + yd = 1, \ \forall s.$

Eg:- $\phi(s) = C(sI - A)^{-1}B + 1) = \dfrac{s+2}{(s+1)(s+4)}$

$= \dfrac{n(s)}{d(s)}$

Definition: $\mathcal{M}(S)$:- matrices in S.

Eg: $D = \begin{bmatrix} \frac{1}{s+1} & \frac{s+3}{(s+4)^2} \end{bmatrix} \in \mathcal{M}(S)$

Co-Prime Matrices: Let $A, B \in \mathcal{M}(S)$

Left Co-prime: Let A, B have same # of rows. A and B are said to be "Left-coprime" if $\exists X, Y \in \mathcal{M}(S)$ s.t. $\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = I$

$\iff$ Rank $[A(s) \ \ B(s)]$ = # of rows, $\forall s$ in closed RHP & ∞.

Right Co-prime: Let A, B have same # of columns. A, B are said to be "Right co-prime" if $\exists X, Y \in \mathcal{M}(S)$ s.t. $\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = I$

$\iff$ Rank $\begin{bmatrix} A(s) \\ B(s) \end{bmatrix}$ = # of cols, $\forall s$ in closed RHP & ∞.

Facts:

a) A, B are left co-prime $\Rightarrow$ A^T, B^T are right co-prime.

b) If A, B are left co-prime & $XA + YB = I$, then X, Y are right coprime.

c) Square matrix $D \in \mathcal{M}(S)$ has an inverse in $\mathcal{M}(S)$ iff $\det(D) \in U$.
We call such matrix D as <u>Unimodular</u> matrix. $\iff \det(D(s)) \neq 0$.

Right Co-prime Factorisation: A pair (N, D) is a right co-prime factorisation (RCF) if $P(s) = N(s)[D(s)]^{-1}$ with N, D being right co-prime.

Left Co-prime Factorisation: A pair $(\tilde{D}, \tilde{N})$ is a left co-prime factorisation (LCF) if $P(s) = [\tilde{D}(s)]^{-1}\tilde{N}(s)$ with $\tilde{N}, \tilde{D}$ being left co-prime.

Intertwining Property: If $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$, then $\boxed{\tilde{D}N = \tilde{N}D}$ always holds true!

Lemma: Let (N, D) and $(\tilde{D}, \tilde{N})$ be RCF and LCF of P resp. Then $\exists\, X, Y, \tilde{X}, \tilde{Y}$ such that

$$\begin{aligned} XN + YD &= I \\ \tilde{N}\tilde{X} + \tilde{D}\tilde{Y} &= I \end{aligned} \iff \begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{D} \end{bmatrix}\begin{bmatrix} D & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = \begin{bmatrix} I & R \\ 0 & I \end{bmatrix},$$

where R is of the form $-Y\tilde{X} + X\tilde{Y}$.

$$\iff \begin{bmatrix} I & -R \\ 0 & I \end{bmatrix}\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{D} \end{bmatrix}\begin{bmatrix} D & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

(by left multiplication of RHS^{-1})

$$\iff \begin{bmatrix} Y' & X' \\ -\tilde{N} & \tilde{D} \end{bmatrix}\begin{bmatrix} D & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Doubly Co-prime Factorisation: $Y'D + X'N = I$ and $Y'\tilde{X} = X'\tilde{Y}$ (need not be true always)

$\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I$

Example: $G(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{2}{s+1} \end{bmatrix} \Rightarrow$ RCF $= \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}^{-1}$

LCF $= [(s+2)(s+1)]^{-1}\begin{bmatrix} s+1 & 2(s+2) \end{bmatrix}$.

Poles & Zeros: Consider SISO transfer function $f(s) = \frac{n(s)}{d(s)}$.

poles of $g(s)$: Values of $s \in \mathbb{C}$, for which $g(s) \to \infty$

zeros of $g(s)$: Values of $s \in \mathbb{C}$, for which $g(s) = 0$.

> **Roots:** Given polynomial $p(s)$, it's roots are obtained by solving $p(s) = 0$

Facts: a) When $n(s)$, $d(s)$ are co-prime, then zeros are roots of $n(s)$ and poles are roots of $d(s)$.

b.) Generalizing to MIMO transfer function $G(s)$, the definitions are

poles of $G(s)$: values of $s \in \mathbb{C}$, for which atleast one entries of $G(s) \to \infty$

Crude Definition for Transmission Zeros: $\text{Rank}(G(s)) = r$, almost $\forall s \in \mathbb{C}$. But for some $s \in \mathbb{C}$, rank drops. Those values are called transmission zeros of $G(s)$.

Characteristic Polynomial of Minimal Realisation, $H(s)$: Also called "Pole polynomial" = LCM of denominators of all possible minors in $H(s)$.

$H(s)$	Pole polynomial
$\begin{bmatrix} \frac{1}{s+1} & 1 \\ 0 & \frac{1}{s+1} \end{bmatrix}$	$(s+1)^2$
$\begin{bmatrix} \frac{1}{s+3} & \frac{1}{s+3} \\ \frac{1}{s+3} & \frac{1}{s+3} \end{bmatrix}$	$(s+3)$

MIMO Zero Location: A rational matrix $H(s)$ of full column rank has a zero @ $s = z_0$ if $\exists$ rational vector $u(s)$ s.t. $u(z_0) \neq 0$, $|u(z_0)| < \infty$ and $\lim_{s \to z_0} [H(s)\, u(s)] = 0$

output vanishes @ $s = z_0$

> Eg: $H(s) = \begin{bmatrix} 1 & 1/s-3 \\ 0 & 1 \end{bmatrix} \Rightarrow$ pole @ $s=3$, zero @ $s=3$??
>
> How? Select $u(s) = \begin{bmatrix} -1 \\ s-3 \end{bmatrix}$ to see $\lim_{s \to 3} [H(s)\, u(s)] = 0$.