4. Controllability & Observability

- Controllability
  - Controllable & Reachable Subspaces, Gramians
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- Observability
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Subspaces of State Space

Consider a CT LDS starting from $x(t_0) = x_0 \in \mathcal{X} \subseteq \mathbb{R}^n$ and its solution

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ \implies x(t) &= \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau. \end{aligned}$$

Interesting Subspaces of $\mathcal{X}$:

1. $\mathfrak{R} \subseteq \mathcal{X}$ that is reachable from origin in **finite** t time steps
2. $\mathfrak{C} \subseteq \mathcal{X}$ from where origin can be reached after **finite** t steps

Then, we design control u_t that takes us to the $x_{\texttt{final}}$. Given a $x_{\texttt{final}}$, check if $\exists u_t$ that takes us from current state to $x_{\texttt{final}}$ in finite time.

Controllable & Reachable Subspaces

Reachable Subspace (Reachable from the origin)

Given $t_f > t_0 \geq 0$, the **reachable subspace** on $[t_0, t_f]$, denoted by $\mathfrak{R}[t_0, t_f]$ consists of all states x_{t_f} for which $\exists u : [t_0, t_f] \to \mathbb{R}^m$ that transfers the state from $x(t_0) = 0$ to $x(t_f) = x_{t_f}$.

$$\mathfrak{R}[t_0, t_f] := \left\{ x_{t_f} \in \mathcal{X} \mid \exists u(\cdot),\, x_{t_f} = \int_{t_0}^{t_f} \Phi(t, \tau) B(\tau) u(\tau) d\tau \right\} \subseteq \mathcal{X}.$$

Controllable Subspace (Controllable to the origin)

Given $t_f > t_0 \geq 0$, the **controllable subspace** on $[t_0, t_f]$, denoted by $\mathfrak{C}[t_0, t_f] \subseteq \mathcal{X}$ consists of all states x_0 for which $\exists u : [t_0, t_f] \to \mathbb{R}^m$ that transfers the state from $x(t_0) = x_0$ to $x(t_f) = 0$.

$$\mathfrak{C}[t_0, t_f] := \left\{ x_0 \in \mathcal{X} \mid \exists u(\cdot),\, 0 = \Phi(t_f, t_0) x_0 + \int_{t_0}^{t_f} \Phi(t, \tau) B(\tau) u(\tau) d\tau \right\}$$

Controllable & Reachable Subspaces

- Matrices $C(t), D(t)$ do not affect computation of $\mathfrak{R}[t_0, t_f], \mathfrak{C}[t_0, t_f]$.
- Parallel connection of similar systems leads to **restricted** reachable and controllable subspaces. For eg. consider the parallel RC network

$$\dot{x} = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ 0 & -\frac{1}{R_2C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix} u$$

  - When two branches have same time constants $\frac{1}{R_1C_1} = \frac{1}{R_2C_2} = \omega$,

$$\mathfrak{R}[t_0, t_f] = \mathfrak{C}[t_0, t_f] = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mid \alpha \in \mathbb{R} \right\}, \quad \forall t_f > t_0 \geq 0.$$

  - However, when they have different time constants $\frac{1}{R_1C_1} \neq \frac{1}{R_2C_2}$,

$$\mathfrak{R}[t_0, t_f] = \mathfrak{C}[t_0, t_f] = \mathbb{R}^2, \quad \forall t_f > t_0 \geq 0.$$

Controllability & Reachability Gramians

Reachability & Controllability Gramians of CT-LTV System

Given $t_f > t_0 \geq 0$, the **reachability & controllability gramians** on $[t_0, t_f]$, denoted by $W_R[t_0, t_f] \succ 0, W_C[t_0, t_f] \succ 0$ resp. are defined as

$$W_R[t_0, t_f] := \int_{t_0}^{t_f} \Phi(t, \tau) B(\tau) B(\tau)^\top \Phi(t, \tau)^\top d\tau,$$

$$W_C[t_0, t_f] := \int_{t_0}^{t_f} \Phi(t_0, \tau) B(\tau) B(\tau)^\top \Phi(t_0, \tau)^\top d\tau.$$

1. Reachability gramian is used to compute $\mathfrak{R}[t_0, t_f]$
2. Controllability gramian is used to compute $\mathfrak{C}[t_0, t_f]$
3. Both gramians give the respective minimum-energy control to perform the required state transfer.

Computing Reachable Subspace from its Gramian

Reachable Subspace from Reachability Gramian

Given $t_f > t_0 \geq 0$, the reachable subspace is given by

$$\mathfrak{R}[t_0, t_f] = \mathtt{Im}\left(W_R[t_0, t_f]\right).$$

Let $v \in \mathbb{R}^n$ and if $x_{t_f} = W_R[t_0, t_f]v \in \mathtt{Im}\left(W_R[t_0, t_f]\right)$, then the **open loop** control input $u^\dagger_{\mathfrak{R}}(t) = B(t)^\top \Phi(t_f, t)^\top v$ for all $t \in [t_0, t_f]$ can be used to transfer the state from $x(t_0) = 0$ to $x(t_f) = x_{t_f}$.

- $\exists$ other $u(t)$ achieving same goal but $u^\dagger_{\mathfrak{R}}(t)$ does it with $\min$ energy.
- When $x_{t_f} = W_R[t_0, t_f]v \in \mathfrak{R}[t_0, t_f]$, the control $u^\dagger_{\mathfrak{R}}(t)$ transfers the state from $x(t_0) = 0$ to $x(t_f) = x_{t_f}$ with $\min$ energy given by

$$\int_{t_0}^{t_f} \|u(\tau)\|^2 \, d\tau = v^\top W_R[t_0, t_f] v = x_{t_f}^\top W_R[t_0, t_f]^{-1} x_{t_f}.$$

Computing Controllable Subspace from its Gramian

Controllable Subspace from Controllability Gramian

Given $t_f > t_0 \geq 0$, the controllable subspace is given by

$$\mathfrak{C}[t_0, t_f] = \mathtt{Im}\left(W_C[t_0, t_f]\right).$$

Let $v \in \mathbb{R}^n$ and if $x_0 = W_C[t_0, t_f]v \in \mathtt{Im}\left(W_C[t_0, t_f]\right)$, then the **open loop** control input $u_{\mathfrak{C}}^{\dagger}(t) = -B(t)^\top \Phi(t_0, t)^\top v$ for all $t \in [t_0, t_f]$ can be used to transfer the state from $x(t_0) = x_0$ to $x(t_f) = 0$.

- $\exists$ other $u(t)$ achieving same goal but $u_{\mathfrak{C}}^{\dagger}(t)$ does it with $\min$ energy.
- When $x_0 = W_C[t_0, t_f]v \in \mathfrak{C}[t_0, t_f]$, the control $u_{\mathfrak{C}}^{\dagger}(t)$ transfers the state from $x(t_0) = x_0$ to $x(t_f) = x_0$ with $\min$ energy given by

$$\int_{t_0}^{t_f} \|u(\tau)\|^2 \, d\tau = v^\top W_C[t_0, t_f] v = x_0^\top W_C[t_0, t_f]^{-1} x_0.$$

Controllability Matrix - LTI Systems

Consider the CT-LTI system, its reachability & controllability gramians.

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n,\, u \in \mathbb{R}^m,$$

$$W_R[t_0, t_f] = \int_{t_0}^{t_f} e^{A(t_f-\tau)} BB^\top e^{A^\top (t_f-\tau)} d\tau = \int_0^{t_f-t_0} e^{At} BB^\top e^{A^\top t} dt$$

$$W_C[t_0, t_f] = \int_{t_0}^{t_f} e^{A(t_0-\tau)} BB^\top e^{A^\top (t_0-\tau)} d\tau = \int_0^{t_f-t_0} e^{-At} BB^\top e^{-A^\top t} dt$$

Connection Between Controllability Matrix & Gramians

Controllability Matrix: $\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times nm}$

$$\mathfrak{R}[t_0, t_f] = \mathtt{Im}\left(W_R[t_0, t_f]\right) = \mathtt{Im}\left(\mathcal{C}\right) = \mathtt{Im}\left(W_C[t_0, t_f]\right) = \mathfrak{C}[t_0, t_f]$$

Controllability Matrix, Subspaces & Gramians

- **Time Reversibility:** For LTI systems $\mathfrak{R}[t_0, t_f] = \mathfrak{C}[t_0, t_f]$. Hence, we can go to a state x_{t_f} from origin and come back to origin from x_{t_f}.
- **Time Scaling:** $\mathfrak{R}[t_0, t_f], \mathfrak{C}[t_0, t_f]$ is independent of interval $[t_0, t_f]$. If a state transfer is possible in $[t_0, t_f]$, then it is also possible in $[\bar{t}_0, \bar{t}_f]$. Hence, we do not generally specify the interval for $\mathfrak{R}[t_0, t_f], \mathfrak{C}[t_0, t_f]$.

For eg., consider the same parallel RC network. Controllability matrix is

$$\mathcal{C} = \begin{bmatrix} \frac{1}{R_1C_1} & -\frac{1}{R_1^2C_1^2} \\ \frac{1}{R_2C_2} & \frac{1}{R_2^2C_2^2} \end{bmatrix} = \begin{bmatrix} \omega & -\omega^2 \\ \omega & -\omega^2 \end{bmatrix}. \quad \text{Then } \forall t_f > t_0 \geq 0,$$

$$\mathfrak{R}[t_0, t_f] = \mathfrak{C}[t_0, t_f] = \mathtt{Im}(C) = \begin{cases} \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mid \alpha \in \mathbb{R} \right\}, & \text{if } \frac{1}{R_1C_1} = \frac{1}{R_2C_2} = \omega \\ \mathbb{R}^2, & \text{if } \frac{1}{R_1C_1} \neq \frac{1}{R_2C_2} \end{cases}$$

Changes in Discrete Time Setting

- DT LDS Solution: $x(t_1) = \Phi(t_1, t_0)x_0 + \sum_{\tau=t_0}^{t_1-1} \Phi(t_1, \tau+1)B(\tau)u(\tau)$
- **Remark:** $W_C[t_0, t_f]$ is well defined only when all matrices $A(t_0), A(t_0+1), \ldots, A(t_1-1)$ are non-singular.
- The minimum energy controls $\forall t \in [t_0, t_f - 1]$ are as follows

$$u_{\mathfrak{R}}^{\dagger}(t) = B(t)^{\top}\Phi(t_f, t+1)^{\top} v \quad \text{and} \quad u_{\mathfrak{C}}^{\dagger}(t) = -B(t)^{\top}\Phi(t_0, t+1)^{\top} v.$$

- For DT-LTI system with A invertible, $\mathfrak{R}[t_0, t_f] = \mathfrak{C}[t_0, t_f]$. Otherwise,

$$\mathfrak{R}[t_0, t_f] = \mathtt{Im}\,(\mathcal{C}) \subset \mathfrak{C}[t_0, t_f]$$

- **Time Scaling:** When $t_f - t_0 < n$, we have $\mathfrak{R}[t_0, t_f] \subset \mathfrak{C}[t_0, t_f]$. Otherwise when $t_f - t_0 \geq n$, they coincide.

Controllable & Reachable Systems

Consider the DT - LTV system with states $x \in \mathbb{R}^n$ and inputs $u \in \mathbb{R}^m$.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad \forall t \geq t_0.$$

Reachable System

Given $t_f > t_0 \geq 0$, the LTV system or simply the pair $(A(\cdot), B(\cdot))$ is called **completely state reachable** on $[t_0, t_f]$ if $\mathfrak{R}[t_0, t_f] = \mathbb{R}^n$, meaning that origin can be transferred to any other state in $\mathbb{R}^n$.

Controllable System

Given $t_f > t_0 \geq 0$, the LTV system or simply the pair $(A(\cdot), B(\cdot))$ is called **completely state controllable** on $[t_0, t_f]$ if $\mathfrak{C}[t_0, t_f] = \mathbb{R}^n$, meaning that any state in $\mathbb{R}^n$ can be transferred to origin.

Controllable & Reachable LTI Systems

- For LTI systems, we know that $\mathfrak{R}[t_0, t_f] = \mathfrak{C}[t_0, t_f] = \texttt{Im}(\mathcal{C}) \subseteq \mathbb{R}^n$.
- For LTI systems to be controllable, we require $\texttt{Im}(\mathcal{C}) = \mathbb{R}^n$.
- Equivalently, LTI System is controllable iff $\texttt{Rank}(\mathcal{C}) = n$.
- In DT, we can have $\texttt{Im}(\mathcal{C}) = \mathfrak{R}[t_0, t_f] \subset \mathbb{R}^n$ & yet $\mathfrak{C}[t_0, t_f] = \mathbb{R}^n$

Eigenvector Test for Controllability

- LTI system is controllable iff $\nexists$ non-zero $\texttt{eigvec}(A^\top)$ in $\texttt{Ker}(B^\top)$.
- LTI system is controllable iff $\nexists \neq \mathbf{0}_n$ left $\texttt{eigvec}(A)$ in left $\texttt{Ker}(B)$.

Popov-Belevitch-Hautus (PBH) Test for Controllability

LTI system is controllable iff

$$\texttt{Rank}\left(\begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) = n, \quad \forall \lambda \in \mathbb{C}.$$

Lyapunov Test for Controllability

Theorem

Assume that A is Schur stable. The LTI system is controllable iff $\exists! W \succ 0$ solution to the Lyapunov equation $AW + WA^\top = -BB^\top$ and the unique solution to the Lyapunov equation is given by

$$W = \int_0^\infty e^{A\tau} BB^\top e^{A^\top \tau} d\tau = \lim_{(t_f - t_0) \to \infty} W_R(t_0, t_f)$$

- In DT, use $AWA^\top - W = -BB^\top$ and $W = \sum_{\tau=0}^{\infty} A^\tau BB^\top (A^\top)^\tau$.
- $\exists$ uncontrollable systems for which origin can be reached in ∞ time. For eg., consider the system $\dot{x} = -x + 0 \cdot u$, which can be transferred to origin in ∞ time. But $W = 0$ & hence system is uncontrollable.

Updated Lyapunov Stability Theorem

Updated Lyapunov Theorem

Following 5 conditions are equivalent for LTI system $\dot{x} = Ax, \quad x \in \mathbb{R}^n$

1. CT LTI system is asymptotically (equivalently exponentially) stable
2. $\lambda_i(A) < 0, \forall i$
3. $\forall Q \succ 0, \exists! P \succ 0$ which solves the following Lyapunov equation

$$A^\top P + PA = -Q, \quad \text{and} \quad P := \int_0^\infty e^{A^\top t} Q e^{At} dt.$$

4. $\exists P \succ 0$ for which $A^\top P + PA \prec 0$.
5. $\forall B$, with (A, B) controllable, $\exists! P \succ 0$ which solves Lyapunov eqn.

$$A^\top P + PA = -BB^\top, \quad \text{and} \quad P := \int_0^\infty e^{A^\top t} BB^\top e^{At} dt.$$

Feedback Stabilisation Via Lyapunov Test

Using the Lyapunov Test for controllability, we can design controllers to asymptotically stabilise the system.

Theorem

When LTI system is controllable, $\forall \mu > 0, \exists u = -Kx$, a state feedback controller that places all eigenvalues of closed loop system $\dot{x} = (A - BK)x$ with their real part being almost $-\mu$. The Lyapunov equation to solve is

$$P(A - BK) + (A - BK)^\top P = -2\mu P, \quad \text{where} \quad K := \frac{1}{2} B^\top P.$$

Invariance with respect to Similarity Transformation

Consider the LTI system and its equivalent system obtained via similarity transformation $z = T^{-1}x$.

$$\dot{x} = Ax + Bu \iff \dot{z} = \bar{A}z + \bar{B}u, \quad \text{where } \bar{A} = T^{-1}AT,\ \bar{B} = T^{-1}B.$$

$$\implies \bar{\mathcal{C}} = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \dots & \bar{A}^{n-1}\bar{B} \end{bmatrix} = \begin{bmatrix} T^{-1}B & T^{-1}AB & \dots & T^{-1}A^{n-1}B \end{bmatrix}$$

$$= T^{-1}\mathcal{C}$$

$\texttt{rank}(\bar{\mathcal{C}}) = \texttt{rank}(T^{-1}\mathcal{C}) = \texttt{rank}(\mathcal{C}), \quad$ as T^{-1} is non-singular.

Theorem

- Controllability is preserved via similarity transformation
- (A, B) is controllable iff $(\bar{A}, \bar{B} = (T^{-1}AT, T^{-1}B))$ is controllable.

Controllable Decomposition

Controllable Decomposition: $[\bar{A}, \bar{B}, \bar{C}, T]$ = $\texttt{ctrbf}(A, B, C)$

For every LTI system, $\exists$ a similarity transformation $z = T^{-1}x = \begin{bmatrix} x_c \\ x_u \end{bmatrix}$

that transforms the original LTI system $\dot{x} = Ax + Bu$ to $\dot{z} = \bar{A}z + \bar{B}u$

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_u \end{bmatrix} = \underbrace{\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}}_{:=\bar{A}=T^{-1}AT} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \underbrace{\begin{bmatrix} B_c \\ 0 \end{bmatrix}}_{:=\bar{B}=T^{-1}B} u \quad \text{and } \bar{C} = CT = \begin{bmatrix} C_c & C_u \end{bmatrix}.$$

- Input u cannot affect the states x_u & the pair (A_c, B_c) is controllable
- Transfer function $= C(sI - A)^{-1}B + D = C_c(sI - A_c)^{-1}B_c + D_c$.
- Controllable subspace of $\dot{z} = \bar{A}z + \bar{B}u$ is $\bar{\mathfrak{C}}[t_0, t_f] = \texttt{Im}\left(\begin{bmatrix} I_{\bar{n}} \\ 0 \end{bmatrix}\right)$.

Stabilisability

Stabilisable Systems

The pair (A, B) is **stabilisable** if it is algebraically equivalent to a system in the standard form for uncontrollable systems with $\bar{n} = n$ or with A_u being a stability matrix

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u, \quad x_c \in \mathbb{R}^{\bar{n}}, x_u \in \mathbb{R}^{n-\bar{n}}$$

$$y = \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + Du.$$

- Any controllable system is stabilisable as $\bar{n} = n$ and $\nexists A_u$.
- Asymptotically stable sys. is stabilisable as A_u, A_c are stable matrices.
- Stabilizability (∞-time version of Controllability): $\lim\limits_{t\to\infty} x_u \to 0$.

Changes from Controllability to Stabilisability

- **Eigenvector Test:** For all eigenvalues of $A^\top$ with $\mathtt{Re}(\lambda) \geq 0$, their corresponding eigenvectors should not be in $\mathtt{ker}(B^\top)$.
- **PBH Test:** LTI system is stabilisable iff $\mathtt{rank}\left(\begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) = n$, for all $\lambda \in \mathbb{C}$ such that $\mathtt{Re}(\lambda) \geq 0$.
- **Lyapunov Test:** LTI system is stabilisable iff $\exists P \succ 0$ solution to Lyapunov inequality $AP + PA^\top - BB^\top \prec 0$.
- Feedback stabilisation based on above Lyapunov test is possible.
- Analogous results exist for discrete time.
- If the system is controllable, $\exists u = Kx$ state feedback control law to place the poles at our desired locations.

Observability - Motivation of Output Feedback

- The full state feedback control law $u = Kx$ is applicable only when you have access to the entire set of states.
- However, if it is possible to reconstruct the system state based on outputs and inputs applied, such control laws can be applied. For eg, a possible choice of reconstruction given $\exists C^{-1}$ is

$$x(t) = C^{-1}(y(t) - Du(t))$$

- In practice $p < n$. Still we can reconstruct x from u, y over an interval $[t_0, t_f]$.

Observability Find $x(t_0)$ from future i/p $u(t)$, o/p $y(t)$, $t \in [t_0, t_f]$.

Constructibility Find $x(t_f)$ from past i/p $u(t)$, o/p $y(t)$, $t \in [t_0, t_f]$.

Unobservable Subspace

Let CT-LTV system starts at $x(t_0) = x_0$ with $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t),$$

$$\implies y(t) = C(t)\Phi(t_f, t_0)x_0 + \int_{t_0}^{t_f} C(t)\Phi(t_f, \tau)B(\tau)u(\tau)d\tau + D(t)u(t).$$

For observability, we look for conditions to solve

$$\bar{y}(t) = C(t)\Phi(t_f, t_0)x_0, \quad \forall t \in [t_0, t_f], \quad \text{where} \tag{1}$$

$$\bar{y}(t) = y(t) - \int_{t_0}^{t_f} C(t)\Phi(t_f, \tau)B(\tau)u(\tau)d\tau - D(t)u(t) \tag{2}$$

Unobservable Subspace

Given $t_f > t_0 \geq 0$, unobservable subspace denoted by $\mathfrak{UO}[t_0, t_f]$ consists of all states $x_0 \in \mathbb{R}^n$ for which $C(t)\Phi(t_f, t_0)x_0 = 0, \forall t \in [t_0, t_f]$.

Properties of Unobservable Subspace

Given $t_f > t_0 \geq 0$ and input output pair $(u(t), y(t)), t \in [t_0, t_f]$

1. If an initial state x_0 is compatible with input output pair (agrees (1)), then every initial state of the form $x_0 + x_u, x_u \in \mathfrak{UO}[t_0, t_f]$ is also compatible with same input output pair. Because,

$$\bar{y}(t) = C(t)\Phi(t_f, t_0)x_0, \quad \text{and } C(t)\Phi(t_f, t_0)x_u = 0, \quad \forall t \in [t_0, t_f]$$

$$\implies \bar{y}(t) = C(t)\Phi(t_f, t_0)(x_0 + x_u), \quad \forall t \in [t_0, t_f].$$

2. If $\mathfrak{UO}[t_0, t_f] = \{0\}$, then $\exists! x_0$ that is compatible with i/p-o/p pair. It is then possible to **uniquely** reconstruct the state from i/p-o/p.

3. Matrices $B(\cdot), D(\cdot)$ do not play a role in defining $\mathfrak{UO}[t_0, t_f]$

Observable System

Given $t_f > t_0 \geq 0$, the CT-LTV system is observable if $\mathfrak{UO}[t_0, t_f] = \{0\}$.

Unconstructible Subspace

Future states $x(t_f) = x_{t_f}$ can be related to i/p-o/p on interval $[t_0, t_f]$.

Unconstructible Subspace

Given $t_f > t_0 \geq 0$, the unconstructible subspace on $[t_0, t_f]$, $\mathfrak{UC}[t_0, t_f]$ consists of all final state x_{t_f} for which $C(t)\Phi(t_0, t_f)x_{t_f} = 0$, $\forall t \in [t_0, t_f]$

- If a final state x_{t_f} is compatible with i/p-o/p pair, then every final state of the form $x_{t_f} + x_u$, $x_u \in \mathfrak{UC}[t_0, t_f]$ is also compatible with same i/p-o/p pair.
- If $\mathfrak{UC}[t_0, t_f] = \{0\}$, then $\exists! x_{t_f}$ final state that is compatible with i/p-o/p pair.

Constructible System

Given $t_f > t_0 \geq 0$, CT-LTV system is constructible if $\mathfrak{UC}[t_0, t_f] = \{0\}$.

Observability & Constructibility Gramians

Given $t_f > t_0 \geq 0$, the observability and constructibility gramians are

$$W_{\mathfrak{O}}[t_0, t_f] = \int_{t_0}^{t_f} \Phi(\tau, t_0)^\top C^\top(\tau) C(\tau) \Phi(\tau, t_0) d\tau$$

$$W_{\mathfrak{C}}[t_0, t_f] = \int_{t_0}^{t_f} \Phi(\tau, t_f)^\top C^\top(\tau) C(\tau) \Phi(\tau, t_f) d\tau$$

Relation Between Gramians & Subspaces

Given $t_f > t_0 \geq 0$, the gramians and subspaces are related as

$$\mathfrak{UO}[t_0, t_f] = \texttt{kernel}\left(W_{\mathfrak{O}}[t_0, t_f]\right), \quad \mathfrak{UC}[t_0, t_f] = \texttt{kernel}\left(W_{\mathfrak{C}}[t_0, t_f]\right)$$

1. LTV system is observable iff $\texttt{rank}\left(W_{\mathfrak{O}}[t_0, t_f]\right) = n$
2. LTV system is constructible iff $\texttt{rank}\left(W_{\mathfrak{C}}[t_0, t_f]\right) = n$

Gramian Based Reconstruction

Suppose that we are given $t_f > t_0 \geq 0$, and the i/p-o/p pair $(u(t), y(t)), t \in [t_0, t_f]$.

1. When LTV system is observable

$$x(t_0) = W_{\mathfrak{O}}[t_0, t_f]^{-1} \int_{t_0}^{t_f} \Phi(t, t_0) C(t)^\top \bar{y}(t) dt.$$

2. When LTV system is constructible

$$x(t_f) = W_{\mathfrak{C}}[t_0, t_f]^{-1} \int_{t_0}^{t_f} \Phi(t, t_f) C(t)^\top \tilde{y}(t) dt, \quad \text{where}$$

$$\tilde{y}(t) = y(t) - \int_{t_f}^{t} C(t)\Phi(t, \tau) B(\tau) u(\tau) d\tau - D(t) u(t), \quad \forall t \in [t_0, t_f].$$

Remarks: Whatever we saw till now in CT has its counterpart in DT.

Controllability-Observability Duality of LTI Systems

Consider the original & its transposed LTI systems

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \dot{\bar{x}} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} A^\top & C^\top \\ B^\top & D^\top \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}.$$

Given $t_f > t_0 \geq 0$, the transposed system is

$$\text{controllable} \iff \mathtt{Rank} \underbrace{\int_{t_0}^{t_f} e^{A^\top(\tau - t_0)} C^\top C e^{A(\tau - t_0)} d\tau}_{:= \bar{W}_C[t_0, t_f]} = n$$

$$\text{observable} \iff \mathtt{Rank} \underbrace{\int_{t_0}^{t_f} e^{A(\tau - t_0)} B B^\top e^{A^\top(\tau - t_0)} d\tau}_{:= \bar{W}_{\mathfrak{O}}[t_0, t_f]} = n$$

1. Original sys is controllable iff transposed sys is observable on $[t_0, t_f]$.
2. Original sys is observable on $[t_0, t_f]$ iff transposed sys is controllable.

Reachability-Constructibility Duality of LTI Systems

Given $t_f > t_0 \geq 0$,

1. Original sys is reachable iff transposed sys is constructible on $[t_0, t_f]$.
2. Original sys is constructible on $[t_0, t_f]$ iff transposed sys is reachable.

- Both time scaling and time-reversibility properties hold here too just as in the case of controllability.
- If one can reconstruct the state from future inputs/outputs then one can also reconstruct it from the past inputs/outputs.
- For LTV system, duality is more complicated as state transition matrix of dual system must be equal to transposed state transition matrix of original system and for LTV systems this is not straight-forward.

Observability Matrix

Consider the LTI System $\dot{x} = Ax, y = Cx$ with $x \in \mathbb{R}^n, y \in \mathbb{R}^p$.

- From duality, (A, C) is observable iff $(A^\top, C^\top)$ is controllable.
- Applying the controllability matrix test to pair $(A^\top, C^\top)$, we see that

$$\mathcal{C} = \begin{bmatrix} C^\top & A^\top C^\top & \dots & (A^\top)^{n-1} C^\top \end{bmatrix} = \mathcal{O}^\top, \quad \text{where}$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{pn \times n}, \quad \text{is the } \textbf{observability matrix}.$$

- **Observability Matrix Test:** LTI system is observable iff

  $\mathtt{rank}(\mathcal{C}) = \mathtt{rank}(\mathcal{O}^\top) = \mathtt{rank}(\mathcal{O}) = n$.

Observability Tests

Eigenvector Test for Observability

- LTI system is observable iff $\nexists$ non-zero $\texttt{eigvec}(A)$ in $\texttt{Ker}(C)$.

Popov-Belevitch-Hautus (PBH) Test for Observability

LTI system is observable iff

$$\texttt{Rank}\left(\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}\right) = n, \quad \forall \lambda \in \mathbb{C}.$$

- Eigenvalues corresponding to eigenvectors of A in $\texttt{Ker}(C)$ are called **unobservable modes**.
- For DT, do the appropriate changes.

Lyapunov Test for Observability

Theorem

Assume that A is Schur stable. The LTI system is observable iff $\exists! W \succ 0$ solution to the Lyapunov equation $A^\top W + WA = -C^\top C$ and the unique solution to the Lyapunov equation is given by

$$W = \int_0^\infty e^{A^\top \tau} C^\top C e^{A\tau} d\tau = \lim_{(t_f - t_0) \to \infty} W_{\mathfrak{O}}[t_0, t_f]$$

Observable Decomposition

Consider LTI system & its equivalent system obtained via similarity transformation $z = T^{-1}x$, where $\bar{A} = T^{-1}AT$, $\bar{B} = T^{-1}B, \bar{C} = CT$.

$$\dot{x} = Ax + Bu \iff \dot{z} = \bar{A}z + \bar{B}u,$$

$$y = Cx + Du \iff y = \bar{C}z + Du,$$

$$\implies \bar{\mathcal{O}} = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T = \mathcal{O}T$$

- Observability is preserved via similarity transformation. Since T^{-1} is non-singular, we have $\mathtt{rank}(\bar{\mathcal{O}}) = \mathtt{rank}(\mathcal{O}T) = \mathtt{rank}(\mathcal{O})$.
- (A, C) is observable iff $(\bar{A}, \bar{C}) = (T^{-1}AT, CT)$ is observable.

Observable Decomposition

For every LTI system, $\exists$ a similarity transformation $z = T^{-1}x = \begin{bmatrix} x_o \\ x_u \end{bmatrix}$

that transforms the original system to the form

$$\begin{bmatrix} A_o & 0 \\ A_{21} & A_u \end{bmatrix} = T^{-1}AT, \quad \begin{bmatrix} B_o & B_u \end{bmatrix} = T^{-1}B, \quad \begin{bmatrix} C_o & 0 \end{bmatrix} = CT$$

for which

1. $\overline{\mathfrak{UO}}[t_0, t_f] = \texttt{Image}\left(\begin{bmatrix} 0 \\ I_{\bar{n}\times\bar{n}} \end{bmatrix}\right)$
2. the pair (A_o, C_o) is observable
3. The x_u component of the state cannot be reconstructed from o/p.

Kalman Decomposition

Suppose choose a similarity transformation $x = T^{-1}x$ where

$T = \begin{bmatrix} V_{co} & V_{c\bar{o}} & V_{\bar{c}o} & V_{\bar{c}\bar{o}} \end{bmatrix}$ such that

- Columns of $V_{c\bar{o}}$ form a basis for $A-$invariant subspace $\mathfrak{C} \cap \mathfrak{UO}$
- Columns of $\begin{bmatrix} V_{co} & V_{c\bar{o}} \end{bmatrix}$ form a basis for $A-$invariant controllable subspace $\mathfrak{C}$ of the pair (A, B).
- Columns of $\begin{bmatrix} V_{c\bar{o}} & V_{\bar{c}\bar{o}} \end{bmatrix}$ form a basis for $A-$invariant unobservable subspace $\mathfrak{UO}$ of the pair (A, C).

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{\times o} & 0 \\ A_{c\times} & A_{c\bar{o}} & A_{\times\times} & A_{\times\bar{o}} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{\bar{c}\times} & A_{\bar{c}\bar{o}} \end{bmatrix} \underbrace{\begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix}}_{:=x} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} x + Du$$

Kalman Decomposition Picture

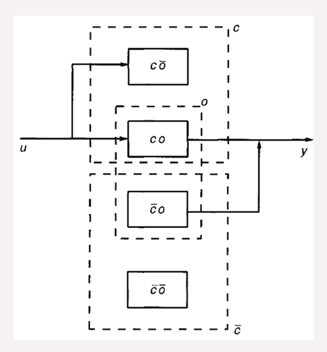

Figure: Schematic representation of canonical Kalman Decomposition

Kalman Decomposition Theorem

For every LTI system, $\exists$ a similarity transformation that transforms it to canonical Kalman Decomposition form for which

1. The pair $\left(\begin{bmatrix} A_{co} & 0 \\ A_{c\times} & A_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \end{bmatrix} \right)$ is controllable
2. The pair $\left(\begin{bmatrix} A_{co} & A_{\times o} \\ 0 & A_{\bar{c}o} \end{bmatrix}, \begin{bmatrix} C_{co} & C_{\bar{c}o} \end{bmatrix} \right)$ is observable
3. The triple (A_{co}, B_{co}, C_{co}) is both controllable & observable
4. Transfer function of the original system is same as the transfer function of the controllable & observable system

$$G(s) = C(sI - A)^{-1}B + D = C_{co}(sI - A_{co})^{-1}B_{co} + D$$

Detectability

Detectable Systems

The pair (A, C) is **detectable** if it is algebraically equivalent to a system in the standard form for unobservable systems with $\bar{n} = n$ or with A_u being a stability matrix

$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ A_{21} & A_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} B_o \\ B_u \end{bmatrix} u, \quad x_o \in \mathbb{R}^{\bar{n}}, x_u \in \mathbb{R}^{n-\bar{n}}$$

$$y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_u \end{bmatrix} + Du.$$

- Any observable system is detectable as $\bar{n} = n$ and $\nexists A_u$.
- Asymptotically stable sys. is detectable as A_u, A_o are stable matrices.

Evolution of Unobservable Components

- The evolution of unobservable component x_u is determined by

$$\dot{x}_u = A_u x_u + \underbrace{A_{21} x_o + B_u u}_{:=v}$$

Consider the new input v to $\dot{x}_u$. Then,

$$x_u(t) = e^{A_u(t_f - t_0)} x_u(t_0) + \int_{t_0}^{t_f} e^{A_u(t_f - \tau)} v(\tau) d\tau$$

- Since the pair (A_o, C_o) is observable, it is possible to reconstruct x_0 from i/p-o/p.
- For detectable systems, the term $e^{A_u(t_f - t_0)} x_u(t_0)$ eventually converges to zero and so we can guess $x_u(t)$ upto an error that converges to zero exponentially fast.

Tests for Detectability

- **Eigenvector Test:** For all eigenvalues of A with $\texttt{Re}(\lambda) \geq 0$, their corresponding eigenvectors should not be in $\texttt{ker}(C)$.
- **PBH Test:** LTI system is detectable iff $\texttt{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = n$, for all $\lambda \in \mathbb{C}$ such that $\texttt{Re}(\lambda) \geq 0$.
- **Lyapunov Test:** LTI system is detectable iff $\exists P \succ 0$ solution to Lyapunov inequality $A^\top P + PA - C^\top C \prec 0$.
- Analogous results exist for discrete time.