Linear Matrix Inequalities in Control

Lecture Notes

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What is a LMI

Linear Matrix Inequalities [1]

A linear matrix inequality (LMI) has the form

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i \succ 0 \tag{1}$$

- $x \in \mathbb{R}^m$ is the variable
- $F_i = F_i^\top \in \mathbb{R}^{n \times n}, i = 0, \ldots, m$ are given symmetric matrices

Facts:

1. LMIs can represent a wide variety of convex constraints on x
2. LMIs help us to formulate matrices as optimization variables
3. Multiple LMIs can be expressed as a single LMI

$$F^{(1)}(x) > 0, \ldots, F^{(p)}(x) > 0 \iff \mathbf{diag}\left(F^{(1)}(x), \ldots, F^{(p)}(x)\right) > 0$$

History of LMI

Wide variety of problems arising in systems & control theory can be reduced to a few standard convex or quasiconvex optimization problems involving LMIs

Lyapunov Theory (1890)

The differential equation

$$\dot{x}(t) = Ax(t)$$

is stable (i.e., all trajectories converge to zero) iff $\exists P = P^\top \succ 0$ such that

$$A^\top P + PA \prec 0$$

Important Timelines

- 1960s - Positive Real Lemma
- 1980s - Interior-point methods for LMIs

What are we learning today?

Flow of Topics

1. Preliminary Topics
2. LMIs for Controllability & Feedback Stabilization
3. LMIs for Observability & Observer Design
4. LMI for H_2-Optimal Full-State Feedback Control
5. LMI for H_∞-Optimal Full-State Feedback Control
6. LMIs for Quadratic Stability with Affine Polytopic & Interval Uncertainty
7. LMIs for Robust Control (Still in Preparation)
8. LMIs in Sum of Squares (SOS) Optimization

How are we learning today?

Learning Steps

1. Study properties about the autonomous system (Eg. $\dot{x} = Ax$ or $x_{k+1} = Ax_k$)
2. Implement a full-state feedback control $u = Kx$
3. Implement an output feedback control $u = K\hat{x}$
4. Study above three with H_2 optimality and H_∞ optimality
5. Study the system with uncertainty (Eg. $\dot{x} = (A + \Delta)x$ or $x_{k+1} = (A + \Delta)x_k$)
6. Implement full-state feedback $u = Kx$ & subsequently output feedback $u = K\hat{x}$
7. Study LMIs for different forms of Δ and design optimal controllers w.r.t H_2, H_∞ norms
8. Miscellaneous LMIs in Sum of Squares Optimization & other problems

Slide Ideas borrowed from [2] and [3]

Preliminary Topics

The Dual Lyapunov LMI

Problem 1

Find $X > 0$ such that

$$A^\top X + XA < 0$$

Problem 2

Find $Y > 0$ such that

$$AY + YA^\top < 0$$

The Dual Lyapunov LMI

Problem 1

Find $X > 0$ such that

$$A^\top X + XA < 0$$

Problem 2

Find $Y > 0$ such that

$$AY + YA^\top < 0$$

Claim: Problem 1) is equivalent to Problem 2).

The Dual Lyapunov LMI

Problem 1

Find $X > 0$ such that

$$A^\top X + XA < 0$$

Problem 2

Find $Y > 0$ such that

$$AY + YA^\top < 0$$

Claim: Problem 1) is equivalent to Problem 2).

Proof: 1) solves 2). Suppose $X > 0$ solves 1). Define $Y = X^{-1} > 0$. Since $A^\top X + XA < 0$, we have

$$X^{-1}\left(A^\top X + XA\right)X^{-1} < 0 \iff X^{-1}A^\top + AX^{-1} < 0 \iff YA^\top + AY < 0$$

Therefore, Problem 2) is feasible with solution $Y = X^{-1}$.

The Dual Lyapunov LMI

Problem 1

Find $X > 0$ such that

$$A^\top X + XA < 0$$

Problem 2

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Proof: 1) solves 2). Suppose $X > 0$ solves 1). Define $Y = X^{-1} > 0$. Since $A^\top X + XA < 0$, we have

$$X^{-1}\left(A^\top X + XA\right)X^{-1} < 0 \iff X^{-1}A^\top + AX^{-1} < 0 \iff YA^\top + AY < 0$$

Therefore, Problem 2) is feasible with solution $Y = X^{-1}$.

Proof: 2) solves 1). Suppose $Y > 0$ solves 2). Define $X = Y^{-1} > 0$. Then

$$A^\top X + XA = X\left(AX^{-1} + X^{-1}A^\top\right)X = X\left(AY + YA^\top\right)X < 0$$

Conclusion: If $V(x) = x^\top Px$ proves stability of $\dot{x} = Ax$, then $V(x) = x^\top P^{-1}x$ proves stability of $\dot{x} = A^\top x$.

LMIs for Controllability & Feedback Stabilization

Continuous & Discrete Time Stability

Guaranteeing Continuous Time Stability

System matrix A is *Hurwitz* iff $\forall Q \succ 0, \exists P \succ 0$ such that $A^\top P + PA = -Q \prec 0$. One such solution is

$$P = \int_0^\infty e^{A^T s} Q e^{As} ds.$$

Guaranteeing Discrete Time Stability

System matrix A is *Schur* iff $\forall Q \succ 0, \exists P \succ 0$ such that $A^\top PA - P = -Q \prec 0$. One such solution is

$$P = \sum_{k=0}^{\infty} (A^T)^k Q A^k.$$

LMI for Controllability Gramian - Continuous Time Case

Consider the state-space system

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \quad x(0) = 0. \end{aligned}$$

Definition

The Controllability Gramian of pair (A, B) is

$$W = \int_0^\infty e^{As} B B^\top e^{A^T s} ds.$$

An LMI for the Controllability Gramian

If (A, B) is *controllable*, then $W \succ 0$ is the unique solution to

$$AW + WA^\top + BB^\top = 0.$$

LMI for Controllability Gramian

Consider the state-space system

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \quad x(0) = 0. \end{aligned}$$

An LMI for the Controllability Gramian

If (A, B) is *controllable*, then $W \succ 0$ is the unique solution to

$$AW + WA^\top + BB^\top = 0.$$

Question: Can we get to any desired state, $x_d(t)$, by using $u(t)$?

LMI for Controllability Gramian

Consider the state-space system

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t), \quad x(0) = 0.$$

Question: Can we get to any desired state, $x_d(t)$, by using $u(t)$?
Answer: The Controllability Gramian tells us which directions are easily controllable and the input $u(t)$ which achieves $x_d(t)$ has the magnitude

$$\|u\|_{L_2}^2 = x_d^\top W_t^{-1} x_d.$$

Caution

- Feasibility of controllability gramian LMI requires A to be stable.
- If A were unstable, some directions would require no energy to reach.

LMI for Stabilizability

- Weaker condition than controllability
- System is stabilizable if uncontrollable subspace is naturally stable.

LMI for Stabilizability

The pair (A, B) is stabilizable iff $\exists X \succ 0, \gamma > 0$ such that

$$AX + XA^\top - \gamma BB^\top \prec 0$$

and the stabilizing control input is $u(t) = -\frac{1}{2}B^\top X^{-1}x(t)$.

Good News

- Feasibility of the stabilizability LMI does NOT require A to be stable
- The stabilizing controller is a feedback gain

The Static State Feedback Problem

The Static State Feedback Problem

Find a feedback matrix $K \in \mathbb{R}^{m \times n}$ such that

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ u(t) &= Kx(t) \end{aligned}$$

is stable.

Look for matrix K such that the closed loop system $\dot{x} = (A + BK)x(t)$ is stable.

RECALL LYAPUNOV LMI !!!

LMI for Static State Feedback Problem

Look for matrix K such that the closed loop system $\dot{x} = (A + BK)x(t)$ is stable.

LMI for Static State Feedback

Find a feedback matrix $K \in \mathbb{R}^{m \times n}$ and $X \succ 0$ such that

$$X(A + BK) + (A + BK)^\top X \prec 0.$$

LMI for Static State Feedback Problem

Look for matrix K such that the closed loop system $\dot{x} = (A + BK)x(t)$ is stable.

LMI for Static State Feedback

Find a feedback matrix $K \in \mathbb{R}^{m \times n}$ and $X \succ 0$ such that

$$X(A + BK) + (A + BK)^\top X \prec 0.$$

Problem: Bilinear in K and X !!!

- Resolving this bilinearity is a quintessential step in the controller synthesis
- Bilinear optimization is not convex
- To convexify the problem, we use a change of variables
- Recall Dual Lyapunov LMI

LMI for Static State Feedback Problem

Problem 1: Find $K \in \mathbb{R}^{m \times n}$ and $X \succ 0$ such that

$$X(A + BK) + (A + BK)^\top X \prec 0.$$

is equivalent to

Problem 2: Find $Z \in \mathbb{R}^{m \times n}$ and $P \succ 0$ such that

$$AP + PA^\top + BZ + Z^\top B^\top \prec 0.$$

LMI for Static State Feedback Problem

Problem 1: Find $K \in \mathbb{R}^{m \times n}$ and $X \succ 0$ such that

$$X(A + BK) + (A + BK)^\top X \prec 0.$$

is equivalent to

Problem 2: Find $Z \in \mathbb{R}^{m \times n}$ and $P \succ 0$ such that

$$AP + PA^\top + BZ + Z^\top B^\top \prec 0.$$

- **Problem 2** has a valid LMI now in variables Z, P
- Solve Problem 2) and recover feedback gain matrix $K = ZP^{-1}$.

LMI for Controllability Gramian - Discrete Time Case

Consider the state-space system

$$x_{k+1} = Ax_k + Bu_k, \quad x_0 = 0.$$

Definition

The Discrete-Time Controllability Gramian of pair (A, B) is

$$W = \sum_{k=0}^{\infty} A^k BB^\top (A^\top)^k.$$

An LMI for the Discrete-Time Controllability Gramian

If (A, B) is *controllable*, then $W \succ 0$ is the unique solution to

$$A^\top WA - W + BB^\top = 0.$$

The Discrete-Time Feedback Stabilization Problem

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The Static State Feedback Problem

Find a feedback matrix $K \in \mathbb{R}^{m \times n}$ such that

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ u_k &= Kx_k \end{aligned}$$

is Schur stable.

Look for matrix K such that the closed loop system $x_{k+1} = (A + BK)x_k$ is stable.

AGAIN RECALL LYAPUNOV LMI !!! link

LMI for Discrete-Time Feedback Stabilization Problem

Look for matrix K such that the closed loop system $x_{k+1} = (A + BK)x_k$ is stable.

LMI(Almost) Discrete-Time Feedback Stabilization

Find a feedback matrix $K \in \mathbb{R}^{m \times n}$ and $P \succ 0$ such that

$$(A + BK)^\top P(A + BK) - P \prec 0.$$

LMI for Discrete-Time Feedback Stabilization Problem

Look for matrix K such that the closed loop system $x_{k+1} = (A + BK)x_k$ is stable.

LMI(Almost) Discrete-Time Feedback Stabilization

Find a feedback matrix $K \in \mathbb{R}^{m \times n}$ and $P \succ 0$ such that

$$(A + BK)^\top P(A + BK) - P \prec 0.$$

Work towards a LMI via small trick

$$\begin{aligned}
&(A + BK)^\top P(A + BK) - P \prec 0 \\
\iff\ & P - (A + BK)^\top P(A + BK) \succ 0 \\
\iff\ & P^{-1} - P^{-1}(A + BK)^\top P(A + BK)P^{-1} \succ 0 \\
\iff\ & \begin{bmatrix} P^{-1} & (A + BK)P^{-1} \\ P^{-1}(A + BK)^\top & P^{-1} \end{bmatrix} \succ 0
\end{aligned}$$

Problem: Bilinear in K and P^{-1} !!!

LMI for Discrete-Time Feedback Stabilization Problem

Again we have two equivalent problems.

Problem 1: Find $K \in \mathbb{R}^{m\times n}$ and $P \succ 0$ such that

$$\begin{bmatrix} P^{-1} & (A+BK)P^{-1} \\ P^{-1}(A+BK)^\top & P^{-1} \end{bmatrix} \succ 0$$

is equivalent to

Problem 2: Find $Z \in \mathbb{R}^{m\times n}$ and $X \succ 0$ such that

$$\begin{bmatrix} X & AX+BZ \\ (AX+BZ)^\top & X \end{bmatrix} \succ 0$$

What did we do ?

- Did variable substitutions $P^{-1} = X$ and $Z = KX$
- **Problem 2** has a valid LMI now in variables Z, X
- Solve Problem 2) and recover feedback gain matrix $K = ZX^{-1}$.

LMI for Discrete-Time Stabilizability

LMI for Discrete-Time Stabilizability

The pair (A, B) is stabilizable iff $\exists P \succ 0$ such that

$$APA^\top - P \prec BB^\top$$

LMIs for Observability & Observer Design

Duality Between Observability & Controllability

Consider the state-space system

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t), \quad x(0) = 0.$$

Observability & Controllability are duals of each other

- We can investigate observability of (A, C) by studying controllability of $(A^\top, C^\top)$
- (A, C) is observable if and only if $(A^\top, C^\top)$ is controllable.

LMI for Observability Gramian

Consider the state-space system

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \quad x(0) = 0. \end{aligned}$$

Definition

The Observability Gramian of pair (A, C) is

$$Y = \int_0^\infty e^{A^T s} C^\top C e^{As} ds.$$

LMI for the Observability Gramian

If (A, C) is *observable*, iff $Y \succ 0$ is the unique solution to

$$YA + A^\top Y + C^\top C = 0.$$

LMI for Observer Synthesis

Consider the state-space system

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t), \quad x(0) = 0.$$

FACT

An observer exists if and only if (A, C) is detectable

LMI for Observer Synthesis

There exists an observer with gain L such that $A + LC$ is stable iff $\exists P \succ 0$ and Z such that

$$A^\top P + PA + C^\top Z + Z^\top C \prec 0,$$

where the observer gain matrix is retrieved as $L = P^{-1}Z^\top$.

LMI for H_2-Optimal Full-State Feedback Control

System H_2 Norm

Consider the system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

System H_2 Norm

For a stable, causal continuous time LTI system with state-space model (A, B, C, D), transfer function $G(s)$, and impulse response $G(t)$, the H_2 norm of G, denoted by $\|G\|_{H_2}$ measures

- The energy of impulse response
- For $\|G\|_{H_2}$ to be finite, need strict causality $\iff D = 0$
- When $x_0 = 0$ and u_t is an unit impulse signal,

$$\|G\|_{H_2}^2 := \int_0^\infty \|G(t)\|_F^2 \, dt = \mathbf{Tr}\left[\int_0^\infty G(t)^\top G(t) dt\right]$$

System H_2 Norm Computation

Recall

- Controllability Gramian $W = \int_0^\infty e^{At} BB^\top e^{A^\top t} dt$ satisfies $AW + WA^\top + BB^\top = 0$
- Observability Gramian $Y = \int_0^\infty e^{A^\top t} C^\top C e^{At} dt$ satisfies $A^\top Y + YA + C^\top C = 0$

Computing H_2 norm is easy via state-space methods with $G(t) = Ce^{At}B$

$$\|G\|_{H_2}^2 := \mathbf{Tr}\left[\int_0^\infty G(t)^\top G(t) dt\right] = \mathbf{Tr}\left[\int_0^\infty B^\top e^{A^\top t} C^\top C e^{At} B dt\right] = \mathbf{Tr}\left[B^\top Y B\right]$$

$$\|G\|_{H_2}^2 = \mathbf{Tr}\left[\int_0^\infty G(t) G(t)^\top dt\right] = \mathbf{Tr}\left[\int_0^\infty Ce^{At} BB^\top e^{A^\top t} C^\top dt\right] = \mathbf{Tr}\left[CWC^\top\right]$$

Takeaways

H_2 norm can be computed easily if Controllability or Observability Gramians are calculated

LMI Characterization of H_2 Norm

H_2 Norm Minimization Problem

Find $X = X^\top \succ 0$ such that

- $\|G(s)\|_{H_2} < \gamma$
- $AX + XA^\top + BB^\top \prec 0$

Equivalently, the solution to the following SDP in variables X, P assures that the A is asymptotically stable and the H_2 norm is atmost $\eta = \gamma^2$.

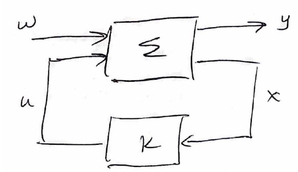
LMI for H_2 Norm Minimization

$$
\begin{aligned}
\underset{\eta, X, P}{\text{minimize}} \quad & \eta \\
\text{subject to} \quad & \mathbf{Tr}(P) < \eta, X \succ 0, P \succ 0 \\
& AX + XA^\top + BB^\top \prec 0 \\
& \begin{bmatrix} P & CX \\ XC^\top & X \end{bmatrix} \succ 0
\end{aligned}
$$

Control Design Using H_2 Norm

Consider the system

$$\dot{x} = Ax + Bu + Fw$$
$$y = Cx + Du$$



Control Design Problem

Design a full state feedback controller $u(t) = Kx(t)$ that stabilizes and minimizes the H_2 norm of the closed loop system from disturbance input w to performance output y.

Control Design Using H_2 Norm

Use the H_2 LMI for closed loop system obtained using full state feedback $u = Kx$

$$\begin{aligned} \dot{x} &= (A + BK)x + Fw \\ y &= (C + DK)x \end{aligned}$$

LMI(Almost) for H_2 Norm Controller Synthesis

$$\begin{aligned} \underset{\eta, X, P}{\text{minimize}} \quad & \eta \\ \text{subject to} \quad & \mathbf{Tr}(P) < \eta, X \succ 0, P \succ 0 \\ & (A + BK)X + X(A + BK)^\top + FF^\top \prec 0 \\ & \begin{bmatrix} P & (C + DK)X \\ X(C + DK)^\top & X \end{bmatrix} \succ 0 \end{aligned} \tag{2}$$

- Bilinear in K, X.
- Let $L = KX$ and solve following SDP in variables η, X, L, P.

LMI for H_2 Norm Controller Synthesis

LMI for Controller Synthesis

$$
\begin{aligned}
\underset{\eta, X, L, P}{\text{minimize}} \quad & \eta \\
\text{subject to} \quad & \mathbf{Tr}(P) < \eta, X \succ 0, P \succ 0 \\
& AX + XA^\top + BL + L^\top B^\top + FF^\top \prec 0 \\
& \begin{bmatrix} P & CX + DL \\ XC^\top + L^\top D^\top & X \end{bmatrix} \succ 0
\end{aligned} \tag{3}
$$

Recover the controller gain as $K = LX^{-1}$

LMI for H_∞-Optimal Full-State Feedback Control

Defining the H_∞ Norm

Consider the proper stable LTI system with transfer function $G(s) = C(sI - A)^{-1}B + D$

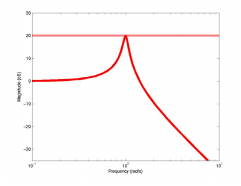
$$\dot{x} = Ax + Bw$$
$$z = Cx + Dw$$

H_∞ Norm

The H_∞ Norm (aka induced L_2 gain) of the above system is given by

$$\|G\|_\infty = \sup_{\|w\|_2 = 1} \|z\|_2$$

It is the worst-case gain of the system



Bounded Real Lemma

Consider the following linear system

$$\dot{x} = Ax + Bu, \quad x(0) = 0,$$
$$y = Cx$$

If a quadratic Lyapunov function $V(x) = x^\top P x$ satisfies

$$\dot{V}(x,u) - \gamma^2 u^\top u + y^\top y \leq 0$$

Then, $\|G\|_\infty \leq \gamma$.

What's the intuition ?

Integrate above inequality & apply boundary conditions to see that $\|G\|_\infty^2 = \frac{\|y\|_2}{\|u\|_2} \leq \gamma^2$

LMI to Compute H_∞ Norm

Consider the following linear system

$$\begin{aligned}\dot{x} &= Ax + Bu, \quad x(0) = 0,\\ y &= Cx\end{aligned}$$

$$\begin{aligned}\text{Then, } \dot{V}(x,u) - \gamma^2 u^\top u + y^\top y \leq 0\\ \iff (Ax+B)^\top Px + x^\top P(Ax+B) - \gamma^2 u^\top u + x^\top C^\top Cx \leq 0\\ \iff \begin{bmatrix} x \\ u \end{bmatrix} \begin{bmatrix} A^\top P + PA + C^\top C & PB \\ B^\top P & \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0, \quad \forall x, u\end{aligned}$$

LMI to Compute H_∞ Norm

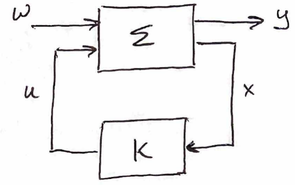
For the above linear system, $\|G\|_\infty \leq \gamma$ iff the following LMI in P is satisfied.

$$\begin{bmatrix} A^\top P + PA + C^\top C & PB \\ B^\top P & \gamma^2 I \end{bmatrix} \preceq 0 \quad \iff \quad \begin{bmatrix} A^\top P + PA & PB & C^\top \\ B^\top P & -\gamma^2 I & 0 \\ C & 0 & -I \end{bmatrix} \preceq 0$$

H_∞ Control Design Problem

Consider the system

$$\dot{x} = Ax + Bu + Fw$$
$$y = Cx + Du$$



H_∞ Control Design Problem

Design a full state feedback controller $u(t) = Kx(t)$ to minimize closed-loop $\|G\|_\infty^2 = \frac{\|y\|_2}{\|w\|_2}$

Trick: Use Bounded Real Lemma for closed-loop with $u = Kx$.

$$\dot{x} = (A + BK)x + Fw$$
$$y = (C + DK)x$$

SDP for H_∞ Control Design

Then, the corresponding LMI that guarantees $\|G\|_\infty^2 = \frac{\|y\|_2}{\|w\|_2} \leq \gamma^2$ is

$$\begin{bmatrix} (A+BK)^\top P + P(A+BK) & PF & (C+DK)^\top \\ F^\top P & -\gamma^2 I & 0 \\ (C+DK) & 0 & -I \end{bmatrix} \preceq 0$$

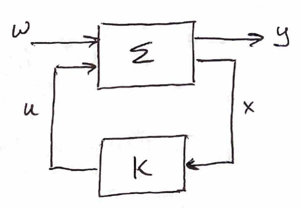
- Bilinear in P, K - Assume $P \succ 0$, let $Q = P^{-1}$. Multiply on left & right by $diag(Q, I, I)$.
- Define variable substitution $L = KQ$ and $\eta = \gamma^2$

SDP for H_∞ Control Design with LMI Constraints

Solve the following SDP & if feasible extract the control gain as $K = LQ^{-1}$.

$$\begin{aligned} \underset{\eta, Q, L}{\text{minimize}} \quad & \eta \\ \text{subject to} \quad & Q \succ 0 \\ & \begin{bmatrix} (AQ+BL) + (AQ+BL)^\top & F & (CQ+DL)^\top \\ F^\top & -\gamma I & 0 \\ CQ+DL & 0 & -\gamma I \end{bmatrix} \preceq 0 \end{aligned}$$

LMIs for Quadratic Stability with Affine Polytopic & Interval Uncertainty



Originally, we solved for K that minimizes the H_∞ norm of the transfer function from w to y.

$$\min_{K \in H_\infty} \|S(\Sigma, K)\|_{H_\infty}$$

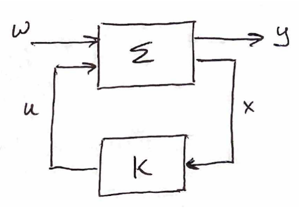
When the system Σ has uncertainty, we have to solve a robust control problem

Robust Control Problem

$$\min_{K \in H_\infty} \gamma : \|S(\Sigma, K)\|_{H_\infty} \leq \gamma, \quad \forall \Sigma \in \mathbf{P}.$$

- $\Sigma \in \mathbf{P}$ is set of all possible plants
- $\mathbf{P}$ can describe either finite or infinite possible systems

Different Types of Modeling Uncertainty



- $\Sigma \in \mathbf{P}$ is set of all possible plants
- $\mathbf{P}$ can describe either finite or infinite possible systems

Set of all possible plants $\mathbf{P}$

The set of all possible plants $\mathbf{P}$ can be characterized as follows

Set of all possible plants $\mathbf{P}$

- Additive Uncertainty: (Focussed Mostly From Now On !!!)

$$\mathbf{P} = \{\Sigma : \Sigma = \Sigma_0 + \Delta, \Delta \in \mathbf{\Delta}\}$$

- Multiplicative Uncertainty:

$$\mathbf{P} = \{\Sigma : \Sigma = (I + \Delta)\Sigma_0, \Delta \in \mathbf{\Delta}\}$$

- Feedback Uncertainty:

$$\mathbf{P} = \left\{\Sigma : \Sigma = \frac{\Sigma_0}{I + \Delta}, \Delta \in \mathbf{\Delta}\right\}$$

- Δ - uncertain system in the uncertainty set $\mathbf{\Delta}$
- Σ_0 - nominal plant (usually known or can be estimated)

Types of Uncertainty - Can be time-varying or time-invariant

- Unstructured, Dynamic, norm-bounded

$$\mathbf{\Delta} := \{\Delta : \| \Delta\|_{H_\infty} < 1\}$$

- Structured, Static, norm-bounded

$$\mathbf{\Delta} := \{diag(\delta_1, \ldots, \delta_k, \Delta_1, \ldots, \Delta_n) : |\delta_i| < 1, \bar{\sigma}(\Delta_i) < 1\}$$

- Structured, Dynamic, norm-bounded

$$\mathbf{\Delta} := \{diag(\Delta_1, \ldots, \Delta_n) : \| \Delta\|_{H_\infty} < 1\}$$

- Unstructured, Parametric, norm-bounded

$$\mathbf{\Delta} := \left\{\Delta \in \mathbb{R}^{n\times n} : \| \Delta\| \leq 1\right\}$$

- Parametric, Polytopic (Simplex)

$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} : \Delta = \sum_i \alpha_i H_i, \alpha_i \geq 0, \sum_i \alpha_i = 1\}$$

- Parametric, Interval

$$\mathbf{\Delta} := \left\{\sum_i \delta_i \Delta_i : \delta_i \in \left[\delta_i^-, \delta_i^+\right]\right\}$$

Stability for Static & Dynamic Uncertainty

Robust Stability for Static Uncertainty

The system

$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$

is Robustly Stable over $\boldsymbol{\Delta}$ if $A_0 + \Delta$ is Hurwitz $\forall \Delta \in \boldsymbol{\Delta}$.

Quadratic Stability for Dynamic Uncertainty

The system

$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$

is Quadratically Stable over $\boldsymbol{\Delta}$ if $\exists P \succ 0$ such that

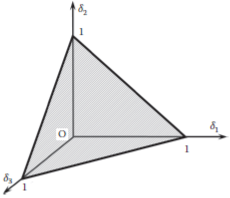
$$(A + \Delta)^\top P + P(A + \Delta) \prec 0, \quad \forall \Delta \in \boldsymbol{\Delta}.$$

- Quadratic Stability - often called “infinite-dimensional LMI” - Hence NOT tractable
- LMI can be made finite for polytopic sets

LMI for Polytopic Uncertainty

Consider the system

$$\dot{x}(t) = (A_0 + \Delta A(t))x(t),$$
$$\Delta A(t) = \sum_{i=1}^{k} A_i \delta_i(t),$$
$$\delta(t) \in \{\delta : \sum_i \alpha_i = 1, \alpha_i \geq 1\}$$



LMI for Polytopic Uncertainty

Above system is quadratically stable over $\mathbf{\Delta} := Co(A_1, \ldots, A_k)$ iff $\exists P \succ 0$ such that

$$(A_0 + A_i)^\top P + P(A_0 + A_i) \prec 0, \quad \text{for } i = 1, \ldots, k.$$

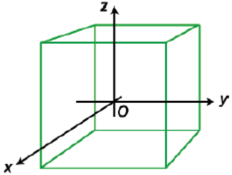
LMI only needs to hold at the VERTICES of the polytope.

LMI for Interval Uncertainty (Kind of Polytopic Uncertainty)

Consider the system

$$\dot{x}(t) = (A_0 + \Delta A(t))x(t),$$

$$\Delta A(t) = \sum_{i=1}^{k} A_i \delta_i(t), \delta_i(t) \in [\delta_i^-, \delta_i^+]$$



The vertices of the hypercube define the vertices of the uncertainty set

$$V := \left\{ A_0 + \sum_{i=1}^{k} A_i \delta_i(t), \delta_i \in [-1, 1] \right\}$$

LMI for Interval Uncertainty

Above system is quadratically stable over $\mathbf{\Delta} := Co(V)$ iff $\exists P \succ 0$ such that

$$\left(A_0 + \sum_{i=1}^{k} A_i \delta_i \right)^{\top} P + P \left(A_0 + \sum_{i=1}^{k} A_i \delta_i \right) \prec 0, \quad \forall \delta \in \{-1, 1\}^k.$$

LMI for Quadratic Polytopic Stabilization

LMI for Quadratic Polytopic Stabilization

There exists a controller gain matrix K such that

$$\dot{x}(t) = (A + \Delta_A + (B + \Delta_B)K)x(t)$$

is quadratically stable for $(\Delta_A, \Delta_B) \in Co((A_1, B_1), \ldots, (A_k, B_k))$ iff $\exists P \succ 0$ and Z such that

$$(A + A_i)P + P(A + A_i)^\top + (B + B_i)Z + Z^\top (B + B_i)^\top \prec 0, \quad i = 1, \ldots, k$$

Controller gain matrix K can be obtained as $K = ZP^{-1}$.

Remarks:

- K is independent of Δ
- Designing $K(\Delta)$ is harder - requires sensing Δ in real time

LMI for Quadratic Polytopic H_∞-Optimal State-Feedback Control

Consider the system

$$\dot{x} = (A + \sum_i A_i)x + (B + \sum_i B_i)u + (F + \sum_i F_i)w$$
$$y = (C + \sum_i C_i)x + (D + \sum_i D_i)u$$

LMI that guarantees $\|G\|_\infty^2 = \frac{\|y\|_2}{\|w\|_2} \leq \gamma^2$ under $u = Kx$ for all $\Delta \in Co(\Delta_1, \ldots, \Delta_k)$ is

SDP for Quadratic Polytopic H_∞-Optimal State-Feedback Control

reference link

Solve the following SDP & if feasible extract the control gain as $K = LQ^{-1}$.

$$\begin{aligned} \min_{\eta,Q,L} \quad & \eta \\ \text{s.t} \quad & Q \succ 0 \\ & \begin{bmatrix} ((A+A_i)Q + (B+B_i)L) + ((A+A_i)Q + (B+B_i)L)^\top & *^\top & *^\top \\ (F+F_i)^\top & -\gamma I & *^\top \\ (C+C_i)Q + (D+D_i)L & 0 & -\gamma I \end{bmatrix} \preceq 0, i = 1:k \end{aligned}$$

LMI for Quadratic Polytopic H_2-Optimal State-Feedback Control

LMI that guarantees $\|G\|_2^2 \leq \gamma^2$ under $u = Kx$ for all $\Delta \in Co(\Delta_1, \ldots, \Delta_k)$ is

SDP for Quadratic Polytopic H_2-Optimal State-Feedback Control reference link

Solve the following SDP & if feasible extract the control gain as $K = LQ^{-1}$.

$$
\begin{aligned}
\min_{\eta, X, L, P} \quad & \eta \\
\text{s.t} \quad & \mathbf{Tr}(P) < \eta, X \succ 0, P \succ 0 \\
& AX + XA^\top + BL + L^\top B^\top + FF^\top + A_i X + X A_i^\top + B_i L + L^\top B_i^\top + F_i F_i^\top \prec 0 \\
& \begin{bmatrix} P & CX + DL \\ XC^\top + L^\top D^\top & X \end{bmatrix} + \begin{bmatrix} 0 & C_i X + D_i L \\ XC_i^\top + L^\top D_i^\top & 0 \end{bmatrix} \succ 0, i = 1, \ldots, k
\end{aligned}
$$

Possible Research: LMI for Quadratic Polytopic H_2-Optimal Output-Feedback Control ???

LMI for Quadratic Schur Stabilization

Consider the system

$$
\begin{aligned}
x_{k+1} &= \left(A + \sum_i A_i\right) x_k + \left(B + \sum_i B_i\right) u_k \\
&= \left(A + \sum_i A_i + \left(B + \sum_i B_i\right) K\right) x_k
\end{aligned}
$$

SDP for Quadratic Schur Stabilization reference link

Suppose $\exists X \succ 0$ and Z such that

$$
\begin{bmatrix} X & AX + BZ \\ XA^\top + Z^\top B^\top & X \end{bmatrix} + \begin{bmatrix} 0 & A_i X + B_i Z \\ X A_i^\top + Z^\top B_i^\top & 0 \end{bmatrix} \succ 0, i = 1, \dots, k
$$

then if $K = ZX^{-1}$, the trajectories of closed loop stable are quadratically stable $\forall \Delta \in Co(\Delta_1, \dots, \Delta_k)$.

LMIs for Robust Control

TO BE DONE !!!

Tentative Topics:

- LMI for Parametric, Norm-Bounded Uncertainty
- LMI for Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty
- LMI for $H_\infty-$Optimal Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty
- LMI for Stability of Structured, Norm-Bounded Uncertainty
- LMI for Stabilizing State-Feedback Controllers with Structured Norm-Bounded Uncertainty
- LMI for $H_\infty-$Optimal State-Feedback Controllers with Structured Norm-Bounded Uncertainty
- D-K Iteration-based Output-Feedback Robust Controller Synthesis

LMIs in Sum of Squares (SOS) Optimization

Polynomial Space & Its Representation

- The set of polynomials is an ∞-dimensional (but Countable) vector space
- Can be made "Finite Dimensional" if we bound the degree
- The monomials form a simple basis for the space of polynomials

Linear Representation of Polynomials

Any polynomial of degree d can be represented as follows

$$p(x) = c^\top B_d(x)$$

- c is vector of coefficients
- $B_d(x)$ is the vector of monomial bases of degree d or less. For instance,

$$B_4(x) = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \end{bmatrix}$$
$$B_2(x_1, x_2) = \begin{bmatrix} 1 & x_1 & x_2 & x_1 x_2 & x_1^2 & x_2^2 \end{bmatrix}$$

LMI for Positive Polynomials

Definition

A polynomial $p(x)$ in $x \in \mathbb{R}^n$ is called Positive Semi-Definite (PSD) if

$$p(x) \geq 0, \quad \forall x \in \mathbb{R}^n.$$

LMI for Positive Polynomials

A polynomial $p(x)$ in $x \in \mathbb{R}^n$ will be PSD $(p(x) \geq 0, \forall x \in \mathbb{R}^n)$ if $\exists P \succeq 0$ such that

$$p(x) = B_d^\top(x) P B_d(x)$$

Proof: If $\exists P \succeq 0$ such that $p(x) = B_d^\top(x) P B_d(x)$, then P can be split as $P = Q^\top Q$. Then,

$$\begin{aligned} p(x) &= B_d^\top(x) P B_d(x) \\ \implies p(x) &= B_d^\top(x) Q^\top Q B_d(x) \\ &= (Q B_d(x))^\top (Q B_d(x)) \\ &= h(x)^\top h(x) \\ &\geq 0 \end{aligned}$$

LMI for Positive Polynomials

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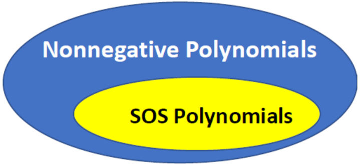
LMI for Positive Polynomials

A polynomial $p(x)$ in $x \in \mathbb{R}^n$ will be PSD $(p(x) \geq 0, \forall x \in \mathbb{R}^n)$ if $\exists P \succeq 0$ such that

$$p(x) = B_d^\top(x) P B_d(x)$$

- We call such polynomials as Sum-of-Squared (SOS), denoted by $p(x) \in \Sigma_s$.
- Equality constraints relate the coefficients of $p(x)$ to the elements of P



Representing Measure of Moments [4]

Given a sequence of moments of an univariate non-negative random variable denoted by

$$\bar{\sigma} = [M_0, M_1, \ldots, M_k].$$

Representing Measure

Does $\bar{\sigma}$ has a representing measure (i.e. probability distribution) μ ?

Representing Measure of Moments [4]

Given a sequence of moments of an univariate non-negative random variable denoted by

$$\bar{\sigma} = [M_0, M_1, \ldots, M_k].$$

Representing Measure

Does $\bar{\sigma}$ has a representing measure (i.e. probability distribution) μ ?

Example

Suppose $\bar{\sigma} = [M_0, M_1, M_2] = [1, 0.5, 0.2]$. Then,

$$var = \mathbb{E}\left[(x - \mathbb{E}[x])^2\right] = M_2 - M_1^2 \geq 0$$

$$\text{But } var = 0.2 - 0.5^2 < 0$$

So, $\bar{\sigma}$ does not have a representing measure

Moments Matrix

Given a sequence of moments of an univariate non-negative random variable denoted by

$$\bar{\sigma} = [M_0, M_1, \ldots, M_k].$$

Representing Measure

Does $\bar{\sigma}$ has a representing measure (i.e. probability distribution) μ ?

LMI Condition on the moments up to order 2

Suppose $\bar{\sigma} = [M_0, M_1, M_2]$ Then,

$$var = \mathbb{E}\left[(x - \mathbb{E}[x])^2\right] = M_2 - M_1^2 \geq 0 \implies \begin{bmatrix} 1 & M_1 \\ M_1 & M_2 \end{bmatrix} \succeq 0$$

Moments Matrix

Moment Matrix associated with $\bar{\sigma}$ up to order $2d$ is the real symmetric square matrix

$$R_d(\bar{\sigma}) = \mathbb{E}_\mu\left[B_d(x)B_d^\top(x)\right].$$

- $B_d(x)$ - vector of monomials up to order d

Moment matrix of order $d = 2$ of a measure in $\mathbb{R}$

The vector of monomials up to order $d = 2$ is $B_2(x) = \begin{bmatrix}1 & x & x^2\end{bmatrix}^\top$. Then,

$$R_2(\bar{\sigma}) = \mathbb{E}_\mu\left[B_2(x)B_2^\top(x)\right] = \begin{bmatrix} M_0 & M_1 & M_2 \\ M_1 & M_2 & M_3 \\ M_2 & M_3 & M_4 \end{bmatrix}$$

- $R_d(\bar{\sigma})$ required moments up to order $2d$
- $R_d(\bar{\sigma}) \in \mathbb{R}^{S_{n,d}\times S_{n,d}}$, where $S_{n,d} = \binom{n+d}{n}$
- Number of moments is $S_{n,2d} = \binom{n+2d}{n}$

LMI Conditions on Moments Matrix

Moments Condition

Moments of every non-negative measure $\mu \in \mathbb{R}^n$ satisfies

$$R_d(\bar{\sigma}) \succeq 0, \quad \forall d.$$

Important Fact

Not every moment sequence $\bar{\sigma}$ that satisfies $R_d(\bar{\sigma}) \succeq 0, \forall d$ has a representing measure $\mu \in \mathbb{R}^n$.

$$(\mu, \bar{\sigma}) \overset{\Rightarrow}{\not\Leftarrow} R_d(\bar{\sigma}) \succeq 0, \forall d.$$

Analogy: Not every non-negative polynomial has a SOS representation

References

S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*. SIAM, 1994.

M. M. Peet, *LMI Methods in Optimal and Robust Control*. Course Notes from ASU, 2018.

T. Summers, *Convex Optimization in Systems & Control*. Course Notes from UTD, 2018.

D. Bertsimas and I. Popescu, "Optimal inequalities in probability theory: A convex optimization approach," *SIAM Journal on Optimization*, vol. 15, no. 3, pp. 780–804, 2005.

Thank you

Any questions ?

Hope you all enjoyed the presentation !