3. Internal Stability & Lyapunov Stability Theorems

- Internal (Lyapunov) Stability
  - Eigenvalue Condition
  - Stability Theorem
- Input-Output BIBO Stability
  - Time Domain Conditions
  - Frequency Domain Conditions
- BIBO vs Lyapunov Stability

Linear Matrix Inequalities (LMIs)

LMI Definition

A linear matrix inequality (LMI) has the form

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i \succ 0 \tag{1}$$

- $x \in \mathbb{R}^m$ is the variable
- $F_i = F_i^\top \in \mathbb{R}^{n \times n}, i = 0, \ldots, m$ are given symmetric matrices

1. LMIs can represent a wide variety of convex constraints on x
2. LMIs help us to formulate matrices as optimization variables
3. Multiple LMIs can be expressed as a single LMI

$$F^{(1)}(x) > 0, \ldots, F^{(p)}(x) > 0 \iff \mathbf{diag}\left(F^{(1)}(x), \ldots, F^{(p)}(x)\right) > 0$$

History of LMI

Many problems arising in systems & control theory can be reduced to a few standard convex or quasiconvex optimization problems with LMIs

Lyapunov Theory (1890)

The differential equation $\dot{x}(t) = Ax(t)$ is stable (i.e., all trajectories converge to zero) iff $\exists P = P^\top \succ 0$ s.t.

$$A^\top P + PA \prec 0$$

Important Timelines

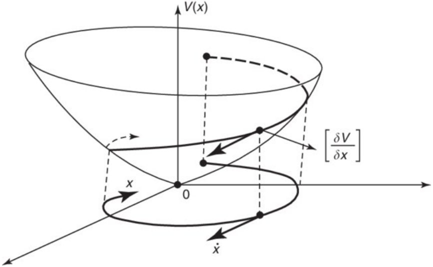
- 1960s - Positive Real Lemma
- 1980s - Interior-point methods for LMIs

Lyapunov Theory

Lyapunov Theory: Investigate about trajectories of $\dot{x} = f(x)$ without solving differential equation (need not compute $\Phi(t, t_0)$).

Lyapunov Theorem (Informal)

If $\exists V : \mathbb{R}^n \to \mathbb{R}$ that satisfies some conditions on V and $\dot{V}$, then trajectories of system satisfy some property. We call such V as the **Lyapunov Function.** V indicates a generalised energy fn. for systems.



Lyapunov Stability of CT-LTV Systems

Consider the CT-LTV system $\dot{x} = A(t)x + B(t)u$, $y(t) = C(t)x + D(t)u$.

Stability in the sense of Lyapunov

- **Marginally Stable:** If $\forall x(t_0) = x_0 \in \mathbb{R}^n$, the homogeneous state response $x(t) = \Phi(t, t_0)x_0$, $\forall t \geq 0$ is uniformly bounded.
- **Asymptotically Stable:** If in addition, $\forall x(t_0) = x_0 \in \mathbb{R}^n$, we have $x(t) \to 0$ as $t \to \infty$.
- **Exponentially Stable:** If in addition, $\forall x(t_0) = x_0 \in \mathbb{R}^n$, $\exists c, \lambda > 0$ such that

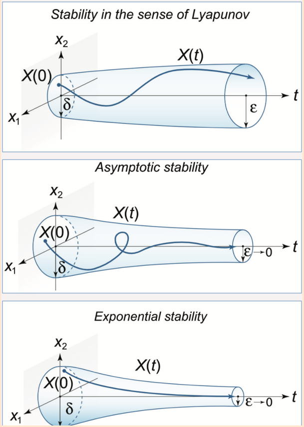
$$\|x(t)\| \leq c e^{-\lambda(t-t_0)} \|x(t_0)\|, \quad \forall t \geq 0$$

- **Unstable:** If it is not marginally stable.

Lyapunov Stability of CT-LTV Systems

Consider the CT-LTV system $\dot{x} = A(t)x$.

Stability in the sense of Lyapunov - Picture



Eigenvalue Conditions for Lyapunov Stability

Eigenvalue Conditions for CT-LTI Systems $\dot{x} = Ax$.

- **Marginally Stable:** Iff $\forall i, \texttt{Re}[\lambda_i] \leq 0$ and all Jordan blocks corresponding to $\texttt{Re}[\lambda_i] = 0$ are 1×1.
- **Asymptotically Stable:** Iff $\forall i, \texttt{Re}[\lambda_i] < 0$.
- **Exponentially Stable:** Iff $\forall i, \texttt{Re}[\lambda_i] < 0$.
- **Unstable:** Iff $\exists i, \texttt{Re}[\lambda_i] > 0$ or the Jordan blocks corresponding to $\texttt{Re}[\lambda_i] = 0$ is larger than 1×1.

Using sub-multiplicative norm property, we have for LTI systems

$$\|x(t)\| = \left\| e^{A(t-t_0)} x_0 \right\| \leq \left\| e^{A(t-t_0)} \right\| \|x_0\| \leq c e^{-\lambda(t-t_0)} \|x_0\|, \quad \forall t \in \mathbb{R}$$

$\Longrightarrow$ "Asymptotic & exponential stability are equivalent concepts"

Lyapunov Stability Theorem

Consider the CT homogeneous LTI system $\dot{x} = Ax, \quad x \in \mathbb{R}^n$.

Lyapunov Theorem

The following 5 conditions are equivalent

1. The CT LTI system is asymptotically stable
2. The CT LTI system is exponentially stable
3. $\lambda_i(A) < 0, \forall i$
4. $\forall Q \succ 0, \exists! P \in \mathbb{S}_n^+$ which solves the following Lyapunov equation

$$A^\top P + PA = -Q$$

5. $\exists P \succ 0$ for which $A^\top P + PA \prec 0$.

Lyapunov Stability of DT-LTV Systems

Let DT-LTV system be

$x(t+1) = A(t)x(t) + B(t)u(t),\ y(t) = C(t)x(t) + D(t)u(t).$

Stability in the sense of Lyapunov

- **Marginally Stable:** If $\forall x(t_0) = x_0 \in \mathbb{R}^n$, the homogeneous state response $x(t) = \Phi(t, t_0)x_0,\ \forall t \geq 0$ is uniformly bounded.
- **Asymptotically Stable:** If in addition, $\forall x(t_0) = x_0 \in \mathbb{R}^n$, we have $x(t) \to 0$ as $t \to \infty$.
- **Exponentially Stable:** If in addition, $\forall x(t_0) = x_0 \in \mathbb{R}^n$, $\exists c > 0, \lambda < 1$ such that $\|x(t)\| \leq c\lambda^{(t-t_0)} \|x(t_0)\|, \quad \forall t \geq 0.$
- **Unstable:** If it is not marginally stable.

Eigenvalue Conditions for Lyapunov Stability

Eigenvalue Conditions for DT-LTI Systems $x(t+1) = Ax(t)$.

- **Marginally Stable:** Iff $\forall i, \mathtt{Re}[\lambda_i] \leq 1$ and all Jordan blocks corresponding to $\mathtt{Re}[\lambda_i] = 1$ are 1×1.
- **Asymptotically Stable:** Iff $\forall i, \mathtt{Re}[\lambda_i] < 1$.
- **Exponentially Stable:** Iff $\forall i, \mathtt{Re}[\lambda_i] < 1$.
- **Unstable:** Iff $\exists i, \mathtt{Re}[\lambda_i] > 1$ or the Jordan blocks corresponding to $\mathtt{Re}[\lambda_i] = 1$ is larger than 1×1.

"A matrix is called **Schur Stable**" if $\forall i, \mathtt{Re}[\lambda_i] < 1$.

Lyapunov Stability Theorem

Consider the DT homogeneous LTI system $x(t+1) = Ax(t), \quad x \in \mathbb{R}^n$.

Lyapunov Theorem

The following 5 conditions are equivalent

1. The DT LTI system is asymptotically stable
2. The DT LTI system is exponentially stable
3. $\lambda_i(A) < 1, \forall i$
4. $\forall Q \succ 0, \exists! P \in \mathbb{S}_n^+$ which solves the following Lyapunov equation

$$A^\top PA - P = -Q$$

5. $\exists P \succ 0$ for which $A^\top PA - P \prec 0$.

Stability of Locally Linearised Systems

- Consider the CT homogenous nonlinear system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ with an equilibrium point $x^\dagger \in \mathbb{R}^n$ such that $f(x^\dagger) = 0$.
- Locally linearising system around the $x^\dagger$ with $\delta x = x - x^\dagger$ results in

$$\dot{\delta x} = \underbrace{\frac{\partial f(x^\dagger)}{\partial x}}_{:=A} \delta x. \tag{2}$$

- Original nonlinear system inherits some of the desirable stability properties of (2)

Theorem for Stability of Linearisation

Assume $f \in \mathbf{C}^2$. If (2) is exponentially stable, then $\exists B_{x^\dagger} \subset \mathbb{R}^n, c, \lambda > 0$ such that

$$\left\|x(t) - x^\dagger\right\| \leq ce^{-\lambda(t-t_0)} \left\|x(t_0) - x^\dagger\right\|, \quad \forall x(t_0) \in B_{x^\dagger}, \forall t \geq t_0. \tag{3}$$

Stability of Locally Linearised Systems

Theorem for Instability of Linearisation

Assume $f \in \mathbf{C}^2$. If (2) is unstable, then $\exists B_{x^\dagger} \subset \mathbb{R}^n, c, \lambda > 0$ such that

$$x(t) \to \infty, \quad \forall x(t_0) \in B_{x^\dagger}, \text{ as } t \to \infty.$$

When linearised system is only marginally stable, the stability of the original nonlinear system **cannot be concluded**. Consider for example,

$$\dot{x} = -x^3 \quad \text{and} \quad \dot{x} = x^3. \tag{4}$$

- Both the systems have the same local linearisation $\dot{\delta x} = 0$ around the equilibrium point $x^\dagger = 0$, which is only marginally stable.
- However, the 1^{st} system is exponentially stable & latter is unstable.

Input-Output Stability

Assume $x_0 = 0$ & study the stability of linear systems under forcing input. Recall for a CT-LTV system, its forced response is given by

$$y_f(t) := \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

BIBO Stability

A CT-LTV system is said to be **uniformly BIBO stable** if $\forall$ bounded $u(t), \exists g \in \mathbb{R}_{\geq 0}$ such that its forced response $y_f(t)$ satisfies

$$\sup_{t\in\mathbb{R}_{\geq 0}} \|y_f(t)\| \leq g \sup_{t\in\mathbb{R}_{\geq 0}} \|u(t)\|$$

- $u(t)$ is **uniformly bounded** if $\exists c < \infty$ such that $\forall t \geq 0, \|u(t)\| \leq c$.
- g corresponds to **system gain** & different norms lead to different g.

Time Domain BIBO Stability Condition

Equivalent Statements for CT LTV Systems

1. The CT LTV system is uniformly BIBO stable
2. All entries of feedthrough matrix $\forall i, j, D_{ij}$ is uniformly bounded and

$$\sup_{t \in \mathbb{R}_{\geq 0}} \int_0^t |g_{ij}(t, \tau)| \, d\tau < \infty$$

   where $g_{ij}(t, \tau)$ marks the entries of $C(t)\Phi(t, \tau)B(\tau)$

Equivalent Statements for CT LTI Systems

1. The CT LTI system is uniformly BIBO stable
2. $\forall i, j, D_{ij}$ are uniformly bounded & $g_{ij}(\rho)$ - entries of $Ce^{A\rho}B$

$$\int_0^\infty |g_{ij}(\rho)| \, d\rho < \infty, \quad \rho = t - \tau$$

BIBO vs Lyapunov Stability

Equivalent Statements - Frequency Domain BIBO Condition

1. The CT(DT) LTI system is uniformly BIBO stable
2. All poles of transfer function $\mathcal{L}[Ce^{At}B](\mathcal{Z}[Ce^{At}B])$ of CT(DT) LTI system have strictly negative real part (magnitude < 1)

Remark: CT LTI system exponentially stable $\underset{\not\Leftarrow}{\Rightarrow}$ it is BIBO stable.

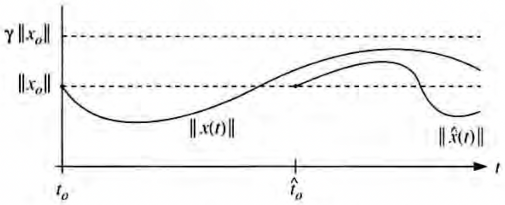
Eg., $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x \implies e^{At} = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$

$\implies$ Lyapunov Unstable. But $Ce^{At}B = e^{-2t} \implies$ BIBO Stable

Similar Results DT-LTV: $y_f(t) := \sum_{\tau=0}^{t-1} C(t)\Phi(t,\tau+1)B(\tau)u(\tau)d\tau + D(t)u(t).$

Let's dive deeper! - Uniform Stability (US)

- For a LTV system, $\dot{x}(t) = A(t)x(t)$, $x(t_0) = x_0$, we are interested in bounds on its solutions that hold regardless of choice of t_0 and x_0.



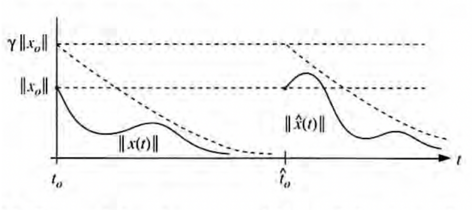
- Study bounds on $\Phi_A(t, t_0)$. But computing $\Phi(t, t_0)$ is not always easy.

Theorem

The LTV system $\dot{x}(t) = A(t)x(t)$ is **uniformly stable** iff $\exists \gamma > 0$ such that $\|\Phi(t, t_0)\| \leq \gamma$, $\forall t \geq t_0$.

Uniformly Exponential Stability (UES)

- For a LTV system, $\dot{x}(t) = A(t)x(t)$, $x(t_0) = x_0$, we impose that for all solutions, $\lim_{t\to\infty} x(t) \to 0$ exponentially.



- Suppose $\exists \alpha > 0$ such that $\forall t, \|A(t)\| \leq \alpha$. Then LTV system is **UES** iff $\exists \beta > 0$ s.t. $\int_{t_0}^{t} \|\Phi(t, t_0)\| \, d\sigma \leq \beta$, $\forall t \geq t_0$ (as $\|\Phi\| \leq 1 + \alpha\beta$).

Theorem

The LTV system $\dot{x}(t) = A(t)x(t)$ is **UES** iff $\exists \gamma, \lambda > 0$ such that $\|\Phi(t, t_0)\| \leq \gamma e^{-\lambda(t-t_0)}$, $\forall t \geq t_0$.

Sometimes, UES is too much to ask

Consider LTV system $\dot{x}(t) = \frac{-2t}{t^2+1}$. Then

$$\Phi(t, t_0) = \frac{t_0^2 + 1}{t^2 + 1} \quad \Longrightarrow \quad \lim_{t \to \infty} \Phi(t, t_0) \to 0, \forall t_0.$$

But, LTV is system is not UES! Suppose, if $\exists \gamma, \lambda > 0$ such that

$$\|\Phi(t, \tau)\| = \frac{\tau^2 + 1}{t^2 + 1} \leq \gamma e^{-\lambda(t-\tau)}, \, \forall t \geq t_0$$

$$\Longrightarrow \; 1 \leq \underbrace{(t^2 + 1)\gamma e^{-\lambda t}}_{:=RHS}, t \geq 0, \quad (\text{setting } \tau = 0)$$

$$\Longrightarrow \; 1 \leq \lim_{t \to \infty} \text{RHS} \to 0, \quad \Longrightarrow \; (\text{contradiction!})$$

What was wrong with the above example ? Solutions went to zero but **not exponentially**.

Uniformly Asymptotic Stability (UAS)

For a LTV system, $\dot{x}(t) = A(t)x(t)$, $x(t_0) = x_0$, is called UAS if it is US and $\forall \delta > 0, \exists T > 0$ such that $\forall t_0, x_0$, the solution satisfies

$$\|x(t)\| \leq \delta \|x_0\|, \quad t \geq t_0 + T.$$

FACT: T is independent of t_0.

Theorem

The LTV system $\dot{x}(t) = A(t)x(t)$ is **UAS** iff it is **UES**.

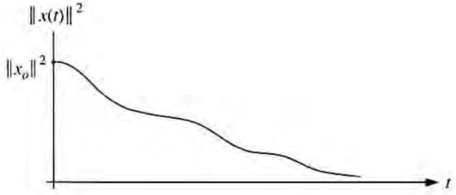
Proof Sketch:

- **UES** $\implies$ **UAS**: For a given $\delta > 0$, pick T such that $e^{-\lambda T} \leq \frac{\delta}{\gamma}$
- **UAS** $\implies$ **UES**: Choose $\delta = \frac{1}{2}$ and x_0 such that $\|x_0\| = 1$

Why Lyapunov Theory?

- Computing State Transition Matrix $\Phi(t, t_0)$ is hard in general.
- **Lyapunov's Idea:** Total energy of an unforced dissipative system decreases as system evolves in time. Stability info revealed by energy-like scalar fns of the states $V(x)$. (finding $V(x)$ is hard!).
- Impose conditions for all solutions of $\dot{x}(t) = A(t)x(t), x(t_0) = x_0$ to monotonically decrease as $t \to \infty$. Let $V(x) := \|x(t)\|^2 = x^\top(t)x(t)$.

$$\implies \dot{V}(x) = \dot{x}^\top(t)x(t) + x^\top(t)\dot{x}(t) = x^\top(t)\left[A^\top(t) + A(t)\right]x(t)$$



Why Lyapunov Theory?

- If $(A^\top(t) + A(t)) \prec 0, \forall t$, then $\|x(t)\|^2$ decreases as $t \to \infty$.
- If $\forall t, \exists \nu > 0$ s.t. $(A^\top(t) + A(t)) \preceq -\nu I$, then $\lim_{t\to\infty} \|x(t)\|^2 \to 0$.
- $V(x) := \|x(t)\|^2_{Q(t)} = x^\top(t)Q(t)x(t)$ is called quadratic Lyapunov function with $Q(t), \forall t$ being symmetric & continuously differentiable.

$$\implies \frac{d}{dt}\left[x^\top(t)Q(t)x(t)\right] = x^\top(t)\left[A^\top(t)Q(t) + Q(t)A(t) + \dot{Q}(t)\right]x(t)$$

- Look for bounds on $V(x), \dot{V}(x)$ for stability. For eg., if $\exists \eta > 0$, then

$$Q(t) \succeq \eta I \quad \iff \quad x^\top(t)Q(t)x(t) \succeq \eta\,\|x(t)\|^2$$

- Study US, UES and Instability using above approach.

Uniform Stability

Uniform Stability - Sufficient Conditions for $\dot{x}(t) = A(t)x(t)$

The above LTV system is **uniformly stable** if $\exists Q(t) \in \mathbb{S}_n, \forall t$ and $Q(t)$ continuously differentiable such that for a given finite constants $\eta, \rho > 0$,

$$\eta I \preceq Q(t) \preceq \rho I, \quad \text{and } A^\top(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \preceq 0$$

$$\implies \|x(t)\| \leq \gamma \|x(t_0)\|, t \geq t_0, \quad \gamma := \sqrt{\frac{\rho}{\eta}} \text{ is independent of } t_0, x(t_0).$$

Eg., Consider the LTV system $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -a(t) \end{bmatrix} x(t), t \geq t_0$, where $a(t)$ is a continuous fn $\forall t$. When $Q(t) = I$, we see that sufficient conditions are satisfied with $\eta = \rho = 1$ and $\dot{V}(x) = \begin{bmatrix} 0 & 0 \\ 0 & -2a(t) \end{bmatrix}$. If $a(t) \geq 0, \forall t$, then LTV system is uniformly stable.

Uniform Exponential Stability (UES)

UES - Sufficient Conditions for $\dot{x}(t) = A(t)x(t)$

The above LTV system is **UES** if $\exists Q(t) \in \mathbb{S}_n, \forall t$ and $Q(t)$ continuously differentiable such that for a given finite constants $\eta, \rho, \nu > 0$,

$$\eta I \preceq Q(t) \preceq \rho I, \quad \text{and } A^\top(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \preceq -\nu I$$

$$\implies \|x(t)\|^2 \leq \frac{\rho}{\eta} e^{-\frac{\nu}{\rho}(t-t_0)} \|x(t_0)\|^2, t \geq t_0, \quad \text{holds } \forall t_0, x(t_0).$$

Though $\exists$ large family of $Q(t)$ matrices, selecting the best $Q(t)$ so that a LTV system is UES is hard. Previous eg., with $Q(t) = I$ is not UES.

Theorem

Suppose LTV system is UES and $\exists \alpha \geq 0$ s.t. $\|A(t)\| \leq \alpha, \forall t$. Then,

$$Q(t) = \int_t^\infty \Phi^\top(\sigma, t)\Phi(\sigma, t)d\sigma \quad \text{satisfies sufficient conditions for UES.}$$

Instability

- Quadratic Lyapunov fns can be used to develop instability criteria
- If $\exists t$, where sign-definiteness of $Q(t)$ is violated, then LTV system would not be uniformly stable.

Instability - Sufficient Conditions for $\dot{x}(t) = A(t)x(t)$

Suppose that $\exists Q(t) \in \mathbb{S}_n, \forall t$ and $Q(t)$ continuously differentiable such that for a given finite constants $\rho, \nu > 0$,

$$\|Q(t)\| \leq \rho, \quad \text{and } A^\top(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \preceq -\nu I.$$

Suppose $\exists t_a$ s.t. $Q(t_a) \nsucceq 0$, then, LTV system is **not uniformly stable**.

Time-Invariant System

UES(UAS) - Sufficient Conditions for $\dot{x}(t) = Ax(t)$

If the system matrix $A \in \mathbb{R}^{n \times n}$ of a LTI system $\dot{x}(t) = Ax(t)$ has negative real-part eigenvalues, then $\forall M \in \mathbb{S}_n, \exists!$ solution $Q \in \mathbb{S}_n$ for the **Lyapunov equation** $A^\top Q + QA = -M$ and Q is given by

$$Q = \int_0^\infty e^{A^\top t} M e^{At} dt, \quad \text{and if } M \succ 0 \implies Q \succ 0.$$

For LTI systems, under same setting involving weaker conditions on M, we can have unique positive definite solution for the Lyapunov Eqn.

Additional Stability Results for LTV Systems

- We saw that $\lambda(A(t))$ are indecisive to infer stability of LTV systems.
- But they do give some info. about growth of solution of LTV systems

Upper & Lower bounds for growth of $\|x(t)\|$ of LTV systems

For a LTV system $\dot{x}(t) = A(t)x(t), t \geq t_0$, denote the largest and smallest point-wise eigenvalues of $A(t) + A^\top(t)$ as $\overline{\lambda}(t), \underline{\lambda}(t)$ respectively. Then, $\forall t_0, x(t_0)$ we have

$$e^{\frac{1}{2}\int_{t_0}^{t}\underline{\lambda}(\sigma)d\sigma}\,\|x(t_0)\| \leq \|x(t)\| \leq e^{\frac{1}{2}\int_{t_0}^{t}\overline{\lambda}(\sigma)d\sigma}\,\|x(t_0)\|\,, \quad t \geq t_0.$$

Proof involves using **Rayleigh-Ritz inequality:** For any real $x \in \mathbb{R}^n$,

$$\underline{\lambda}x^\top x \leq x^\top Q x \leq \overline{\lambda}x^\top x, \quad \text{where } \underline{\lambda}(\overline{\lambda}) = \min(\max)\{\lambda(Q)\}.$$

Conservative Stability of LTV Systems Via Eigenvalues

Uniform Stability (US)

The LTV system $\dot{x}(t) = A(t)x(t), t \geq t_0$ is uniformly stable if $\exists \gamma \in \mathbb{R}$ such that the $\overline{\lambda}(A^\top(t) + A(t))$ satisfies

$$\int_\tau^t \overline{\lambda}(\sigma) d\sigma \leq \gamma, \quad \forall t, \tau \text{ and } t \geq \tau.$$

Uniform Exponential Stability (UES)

The LTV system $\dot{x}(t) = A(t)x(t), t \geq t_0$ is uniformly exponentially stable if $\exists \gamma \in \mathbb{R}, \beta > 0$ such that the $\overline{\lambda}(A^\top(t) + A(t))$ satisfies

$$\int_\tau^t \overline{\lambda}(\sigma) d\sigma \leq \gamma - \beta(t - \tau), \quad \forall t, \tau \text{ and } t \geq \tau.$$

Many LTV systems don't satisfy US/UES condition (very conservative!).

Stability of Perturbed LTV Systems

Study perturbed LTV systems $\dot{z}(t) = [A(t) + F(t)]\, z(t)$ that are close in some sense to $\dot{x}(t) = A(t)x(t)$ which has certain stability property.

Uniform Stability (US)

Suppose that a LTV system $\dot{x}(t) = A(t)x(t), t \geq t_0$ is uniformly stable. Then, the perturbed LTV system $\dot{z}(t) = [A(t) + F(t)]\, z(t), t \geq t_0$ is also uniformly stable if $\exists \beta \in \mathbb{R}$ such that $\forall \tau$

$$\int_{\tau}^{\infty} \|F(\sigma)\|\, d\sigma \leq \beta.$$

Uniform Exponential Stability (UES)

Suppose that LTV system $\dot{x}(t) = A(t)x(t), t \geq t_0$ is UES and $\exists \alpha \geq 0$ such that $\|A(t)\| \leq \alpha$. Then, $\exists \beta \in \mathbb{R}$ such that the perturbed LTV system $\dot{z}(t) = [A(t) + F(t)]\, z(t), t \geq t_0$ is UES if $\|F(t)\| \leq \beta$.

Stability of Slowly Varying LTV Systems

Definition: Slowly Varying LTV Systems

A LTV system $\dot{x}(t) = A(t)x(t), t \geq t_0$ with $A(t)$ continuously differentiable, $\exists \alpha, \mu \geq 0$ such that $\|A(t)\| \leq \alpha, \forall t$ and every point-wise eigenvalues of $A(t)$ satisfying $\mathtt{Re}[\lambda(t)] \leq -\mu$ is referred as **slowly varying system**.

UES of Slowly Varying LTV Systems

Suppose that LTV system $\dot{x}(t) = A(t)x(t), t \geq t_0$ is a slowly varying LTV system. Then, $\exists \beta \geq 0$ such that if the time derivative of $A(t)$ satisfies $\left\|\dot{A}(t)\right\| \leq \beta, \forall t$, the slowly varying LTV system is UES.

Reference: C. Desoer, "Slowly varying system $\dot{x} = A(t)x$", IEEE TAC, 1969. `https://ieeexplore.ieee.org/document/1099336`

DT Changes for Stability Analysis

- Explanations smoothly carry over from CT to DT Lyapunov Stability.
- Use $\Phi(k, k_0)$ instead of $\Phi(t, t_0)$
- Use $V(k) = x^\top(k)Q(k)x(k)$ instead of $V(t) = x^\top(t)Q(t)x(t)$. Subsequently, looking for conditions guaranteeing $\dot{V}(t) < 0$ should change to $V(k+1) - V(k) < 0$.
- Use the DT Lyapunov Equation $A^\top(k)Q(k+1)A(k) - Q(k) \prec 0$ instead of its CT counterpart $QA^\top + AQ \prec 0$.
- Bounds of the form "$\leq e^{-\lambda t}, \lambda < 0$" in the CT will change to "$\leq \lambda^k, \lambda < 1$".

BIBO vs Lyapunov Stability - DT LTV Systems

Impulse Response of DT LTV system: $g(k, k_0) = C(k)\Phi(k, k_0 + 1)B(k_0)$

UES & Uniform BIBO stability of DT LTV Systems

Suppose that a DT LTV system is

1. Uniform BIBO stable $\iff \exists\rho$ s.t. $\sum_{i=k_0}^{k-1} \|g(k,i)\| < \rho, \forall k \geq k_0 + 1$.
2. Assume $A(k), B(k), C(k)$ are bounded and both controller and observer Gramians (to be discussed in the next lecture) satisfy the following inequalities for some $\epsilon > 0$ and $l \in \mathbb{Z}$.

$$\mathcal{C}(k-l, k) \succeq \epsilon I, \quad \mathcal{O}(k, k+l) \succeq \epsilon I.$$

Then, DT-LTV System is UES $\iff$ uniform BIBO stability

Why eg., was Lyapunov Unstable in past slide? Controllability was lost!

Lyapunov Transformation

*"Are stability properties of LTV system preserved under state variable changes ? **NO** in general. But Lyapunov Transformations do!"*

Definition: A $P(t) \in \mathbb{R}^{n\times n}$ that is continuously differentiable & $\exists P^{-1}(t) \forall t$ is called a **Lyapunov Transformation**, if $\exists$ finite $\rho, \eta > 0$ s.t.

$$\|P(t)\| \le \rho, \quad |\mathtt{det}[P(t)]| \ge \eta, \quad \forall t.$$

Stability Preserved Under Lyapunov Transformation

Suppose that $P(t) \in \mathbb{R}^{n\times n}$ is a **Lyapunov Transformation**. Then the LTV system $\dot{x}(t) = A(t)x(t), t \ge t_0$ is US (UES) iff under the variable change $z(t) = P^{-1}(t)x(t)$, the transformed state equation

$$\dot{z}(t) = \left[P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t)\right] z(t) \quad \text{is US(UES).}$$