Exercise Session - 1
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Intent of this exercise

The motive of this exercise session is try out some problems in linear algebra to get a good grasp of that content. This specific exercise session will aide the students to successfully complete the Hand-in # 1. The following exercise problems are suggested for students to be solved.

Problem 1
Compute e^{At} for the following A matrices.

1. $A = \begin{bmatrix} 4 & -2 & 1 \\ 12 & -6 & 3 \\ 8 & -4 & 2 \end{bmatrix}$
2. $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$
3. $A = uv^\top, \quad \forall u, v \in \mathbb{R}^n$

Problem 2
Hypothetically, lets suppose that the dynamics of Sweden-Denmark economy systems be modelled as follows with $S(t)$ and $D(t)$ denoting the states of the two countries respectively,

$$\ddot{S}(t) + \beta_1(t)\dot{S}(t) - \beta_2(t)\dot{D}(t) = u(t)$$
$$\dot{D}(t) = u(t) - D(t) - \beta_3(t)S(t)$$

where $u(t)$ is the control input (cash flow) and $\beta_1(t), \beta_2(t), \beta_3(t) > 0$ are some known constants. Derive the state-space representation of this hypothetical Sweden-Denmark economy dynamical system.

Problem 3
Given two systems defined as follows,

$$\Sigma_A := \quad y_A(t) = \frac{du_A(t)}{dt}$$
$$\Sigma_B := \quad y_B(t) = \alpha(t) \cdot u_B(t)$$

where $y_i(t), u_i(t)$ denote the output and input of systems $i = 1, 2$ respectively and $\alpha(t) > 0$ is a constant. Now consider the new system Σ formed by cascading Σ_A, Σ_B

in series. Determine if the new system Σ linear and time invariant?

Problem 4
Let us investigate the Covid spread in Lund. It has become a folklore knowledge that despite being vaccinated, we are always susceptible to Covid and vaccination will just prevent us from getting into serious troubles. The spread of Covid can be modelled as Susceptible-Infectious-Susceptible (SIS) model from the epidemics literature. Let us denote by $S(t), I(t), N(t)$ the number of people that are susceptible, the number of people that are infected and total number of people in Lund at time t respectively such that

$$N(t) = S(t) + I(t).$$

The mayor of Lund gave an estimate of $N(t) = 92,000$ approximately right now at Lund in 2022. The dynamics of a simplified SIS model with α, β denoting the contact rate, transmission rate respectively can be written as follows

$$\frac{dS(t)}{dt} = -\frac{\alpha S(t)I(t)}{N(t)} + \beta I(t)$$
$$\frac{dI(t)}{dt} = \frac{\alpha S(t)I(t)}{N(t)} - \beta I(t)$$

1. Given the above two-states dynamics, reduce it to just one state $I(t)$ such that

   $$I(t) = f(I(t), N(t), \alpha, \beta).$$

2. Determine the nontrivial equilibrium point ($I_*(t) \neq 0$) of the above reduced order nonlinear Covid dynamics.
3. Compute the linearised dynamics of above reduced order nonlinear Covid dynamics around the nontrivial equilibrium point $I_*(t)$. Determine the stability of the resulting linear system based on the values of α and β and $N(t)$. Comment your inference in words. Based on your inference on the current $N(t)$ and the α, β, will you recommend the Mayor of Lund to impose a lockdown in Lund?

Problem 5
Consider the new system Σ formed by cascading Σ_A, Σ_B in series, where Σ_A is a LTI system, while Σ_B is LTV. Determine if the new system Σ would be linear and time invariant?

Problem 6
Consider the space $\mathcal{H} = L^2[0, 1]$ equipped with the standard inner product

$$\langle f(x), g(x)\rangle = \int_0^1 \bar{f}(x)g(x)dx,$$

where $\bar{f}(x)$ denotes the conjugate of $f(x)$. Define the operator $A : \mathcal{H} \to \mathcal{H}$ as $(Af)(x) = xf(x)$.

1. Show that A is a bounded self-adjoint linear operator on $\mathcal{H}$ and find $\|A\|$.

2. Determine the `Kernel`(A) and `Range`(A).