2. State Space Representations & Linearization

- Autonomous Dynamical Systems
- State Space Representation
- Linearization of Nonlinear Systems
  1. Around Equilibrium Point
  2. Around Trajectory
- Impulse Response, Transfer Functions & Realisation Theory
- Solutions to LTV, LTI & LTP Systems

Representation of Dynamical Systems

System Representation

- Representation refers to a **mathematical model** of a system.
- System representation can be continuous, discrete or hybrid

The **continuous** model of a system starting from $x(0) = x_0$ with $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^p$ is given by

$$\dot{x} = f(x, u, t), \quad y = g(x, u, t)$$

The **discrete** model of a system with $f : \mathbb{R}^n \times \mathbb{R}^m \times \{nT, n \in \mathbb{Z}\} \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \{nT, n \in \mathbb{Z}\} \to \mathbb{R}^p$, time step $T > 0$ and $x(0) = x_0$ is

$$x_{k+1} = f(x_k, u_k, k), \quad y_k = g(x_k, u_k, k)$$

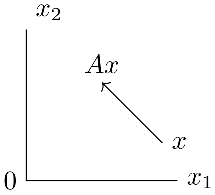
Hybrid Model of a system is given by $\dot{x} = f(x, u, t), \quad y_k = g(x_k, u_k, k)$

Autonomous Dynamical Systems

Autonomous System Representation

Systems **do not have** a forcing i/p "u", but evolve autonomously. That is, $\dot{x}(t) = f(x(t))$.

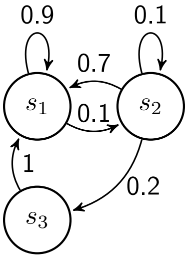
Autonomous LDS is given by $\dot{x}(t) = Ax(t)$.



Invariant Set $\mathcal{S}$: Once trajectory enters $\mathcal{S}$, it stays in $\mathcal{S}$ forever

$\mathcal{S} \subset \mathbb{R}^n$ is **invariant** under $\dot{x} = Ax$ if $\forall x(t) \in \mathcal{S} \implies x(\tau) \in \mathcal{S}$, $\forall \tau \geq t$.

Markov Chain - Autonomous System Example



s_1 means system is okay

s_2 means system is in failure

s_3 means system is under maintenance

$$s(t+1) = \begin{bmatrix} 0.9 & 0.7 & 1 \\ 0.1 & 0.1 & 0 \\ 0 & 0.2 & 0 \end{bmatrix} s(t)$$

Linear Dynamical Systems (LDS)

Continuous Time (CT) LDS

Continuous-time linear dynamical system (CT LDS) has the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

- $t \in \mathbb{R}$ denotes time
- $x(t) \in \mathbb{R}^n$ denotes the state, $\mathbb{R}^n$ denotes the state space $(\mathcal{X})$
- $u(t) \in \mathbb{R}^m$ is the control input, $\mathbb{R}^m$ denotes the input space $(\mathcal{U})$
- $y(t) \in \mathbb{R}^p$ is the output, $\mathbb{R}^p$ denotes the output space $(\mathcal{Y})$
- $A(t) \in \mathbb{R}^{n\times n}, B(t) \in \mathbb{R}^{n\times m}$ are dynamics matrix, input matrix
- $C(t) \in \mathbb{R}^{p\times n}, D(t) \in \mathbb{R}^{p\times m}$ are sensor matrix, feedthrough matrix

Time Invariant Dynamical Systems

The shift operator $\mathcal{T}_\tau : \mathcal{U} \to \mathcal{U}$ is defined as $(\mathcal{T}_\tau u)(t) = u(t - \tau)$

Time Invariant Dynamical System

A dynamical system is said to be **time-invariant** if

1. $\mathcal{U}$ is closed under $\mathcal{T}_\tau, \forall \tau$
2. $\forall t, \tau, \forall x_0, \forall u \in \mathcal{U}, y = g(x, u, t) = g(f(x_0, \mathcal{T}_\tau u, t + \tau), \mathcal{T}_\tau u, t + \tau)$

Linear Time-Invariant (LTI) system can be represented as

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

Interpretation

- Ax: Drift term of $\dot{x}$
- Bu: Input term of $\dot{x}$

Time Varying Dynamical Systems

Time Varying Dynamical System

A dynamical system is said to be **time-varying** if the system dynamics and the output response are parametrised by time.

$$\forall x_0, \forall u \in \mathcal{U},\ \exists t, \tau,\ \text{s.t } y = g(x,u,t) \neq g(f(x_0, \mathcal{T}_\tau u, t+\tau), \mathcal{T}_\tau u, t+\tau)$$

Linear Time-Varying (LTV) system can be represented as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t)$$

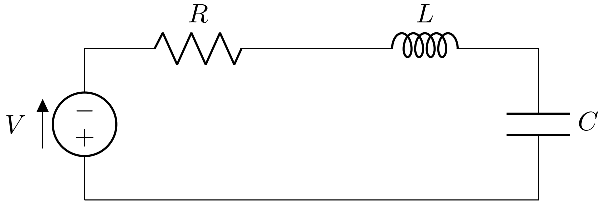
Remarks

- $A(t), B(t), C(t), D(t)$ matrices change over time
- $\{\texttt{LTV Systems}\} \supset \{\texttt{LTI Systems}\}$

Examples of LTI Systems

Series RLC Circuit - LTI System

States: Current via inductor $i(t)$, voltage across capacitor $v_c(t)$



Kirchoff Law: $V(t) = Ri(t) + L\frac{di(t)}{dt} + v_c(t)$, where $v_c(t) = \frac{1}{C}\int i(t)dt$

$$\begin{bmatrix} \frac{di(t)}{dt} \\ \frac{dv_c(t)}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}}_{x} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V(t), \quad Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

Linearisation Near Equilibrium Point

Consider the nonlinear TI differential equation $\dot{x} = f(x, u), y = g(x, u)$.

Equilibrium Point

A pair (x^*, u^*) is called an equilibrium point of the system $\dot{x} = f(x, u)$ if $f(x^*, u^*) = 0$. Once at equilibrium, system stops evolving.

Let $u = u^* + \delta u, x = x^* + \delta x$. Then $y = y^* + \delta y$. So, $\delta x, \delta y$ evolve as

$$\delta\dot{x} = \dot{x} = f(x, u) = f(x^* + \delta x, u^* + \delta u)$$

$$\delta y = y - y^* = g(x, u) - g(x^*, u^*) = g(x^* + \delta x, u^* + \delta u) - g(x^*, u^*)$$

Use Taylor expansions of $f(\cdot), g(\cdot)$ and truncate after 1st order terms

$$\delta\dot{x} \approx \underbrace{\frac{\partial f(x^*, u^*)}{\partial x}}_{:=A} \delta x + \underbrace{\frac{\partial f(x^*, u^*)}{\partial u}}_{:=B} \delta u, \quad \delta y \approx \underbrace{\frac{\partial g(x^*, u^*)}{\partial x}}_{:=C} \delta x + \underbrace{\frac{\partial g(x^*, u^*)}{\partial u}}_{:=D} \delta u$$

Pitfalls of Linearisation Near Equilibrium Point

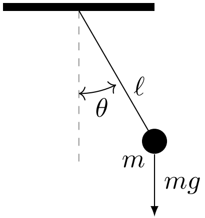
Remarks:

- Linearisation is valid as long as $\delta x, \delta u$ remain small
- Linearised system **not always** gives a good idea of the system behaviour near x^*. For e.g.,
  1. $\dot{x} = -x^3$ near $x^* = 0$ with $x(0) > 0 \implies$ solution is $x(t) = \left(x(0)^{-2} + 2t\right)^{-\frac{1}{2}}$ and linearised system is $\delta\dot{x} = 0$
  2. $\dot{x} = x^3$ near $x^* = 0$ with $x(0) > 0 \implies$ solution is $x(t) = \left(x(0)^{-2} - 2t\right)^{-\frac{1}{2}}$ and linearised system is $\delta\dot{x} = 0$ but has finite escape time at $t = x(0)^{-2}/2$
- To be precise, the above procedure is referred as **local linearisation** of system around an equilibrium point.
- Similar procedure exists for discrete time models

Linearisation of Pendulum

Nonlinear differential equation governing pendulum dynamics is

$$ml^2\ddot{\theta} = -mgl\sin(\theta)$$



If states are $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \implies \dot{x} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) \end{bmatrix}$

Consider the pendulum down $(x = 0)$ equilibrium point. Then the linearised system near $x^* = 0$ is $\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \delta x$

Linearisation Along Trajectory

Perturb around an arbitrary solution of system instead of an eqlb. pt.

Arbitrary Solutions of $\dot{x} = f(x,u), y = g(x,u)$

Suppose $x_{tr} : \mathbb{R}_+ \to \mathbb{R}^n, u_{tr} : \mathbb{R}_+ \to \mathbb{R}^m, y_{tr} : \mathbb{R}_+ \to \mathbb{R}^p$ results in

$\dot{x}_{tr}(t) = f(x_{tr}(t), u_{tr}(t), t), \quad y_{tr}(t) = g(x_{tr}(t), u_{tr}(t), t)$.

$$u(t) = u_{tr}(t) + \delta u(t), x(t) = x_{tr}(t) + \delta x(t) \implies y(t) = y_{tr}(t) + \delta y(t).$$

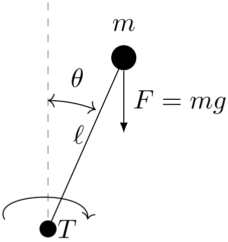
Use Taylor expansions of $f(\cdot), g(\cdot)$ and truncate after 1st order terms

$$\delta\dot{x}(t) \approx \underbrace{\frac{\partial f(x_{tr}(t), u_{tr}(t))}{\partial x}}_{:=A(t)} \delta x(t) + \underbrace{\frac{\partial f(x_{tr}(t), u_{tr}(t))}{\partial u}}_{:=B(t)} \delta u(t),$$

$$\delta y(t) \approx \underbrace{\frac{\partial g(x_{tr}(t), u_{tr}(t))}{\partial x}}_{:=C(t)} \delta x(t) + \underbrace{\frac{\partial g(x_{tr}(t), u_{tr}(t))}{\partial u}}_{:=D(t)} \delta u(t)$$

Remark: Linearisation along trajectory leads to LTV systems

Pendulum Example - Linearisation Along Trajectory

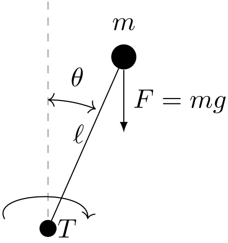


Dynamics: $ml^2\ddot{\theta} = mgl\sin(\theta) - b\dot{\theta} + T$. Linearise around constant angular velocity trajectory $\dot{\theta} = \omega$, with $\theta(0) = 0$. Then

$$\begin{cases} x_1 = \theta \\ x_2 = \dot{\theta} \end{cases} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = T + g\sin(x_1) - x_2 \end{cases}$$

Constant Angular Velocity Trajectory: $\begin{cases} x_1^{sol}(t) = \omega t + x_1(0) = \omega t \\ x_2^{sol}(t) = \omega \end{cases}$

Pendulum Example - Linearisation Along Trajectory



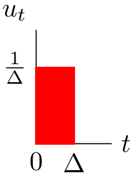
$(x_1^{sol}(t), x_2^{sol}(t))$ should satisfy pendulum equation of motion

$$\begin{cases} \dot{x}_1^{sol} = x_2^{sol} \\ \dot{x}_2^{sol}(t) = T^{sol} + g\sin(x_1^{sol}) - x_2^{sol} \end{cases} \implies T^{sol} = -g\sin(\omega t) + \omega$$

$$A(t) = \frac{\partial f}{\partial x}(x_1^{sol}(t), x_2^{sol}(t)) = \begin{bmatrix} 0 & 1 \\ g\cos(\omega t) & -1 \end{bmatrix}, B(t) = \frac{\partial f}{\partial u}(\cdot,\cdot) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Impulse Response

Impulse signal is a pulse of zero length $(\Delta \to 0)$ but unit area.



CT Impulse Response with m inputs & p outputs

CT Impulse Response is a matrix valued signal $G(t,\tau) \in \mathbb{R}^{p\times m}$ such that $\forall u$, a corresponding output is given by

$$y(t) = \int_0^\infty G(t,\tau)u(\tau)d\tau := (G \star u)(t), \quad \forall t \geq 0.$$

$$\implies y(s) = \mathcal{L}\{y(t)\} = \mathcal{L}\{(G \star u)(t)\} = G(s)u(s), \quad s \in \mathbb{C}$$

Impulse Response of a CT LTI System

Consider the continuous time LTI System

$$\dot{x} = Ax + Bu \implies \mathcal{L}\{\dot{x}\} = \mathcal{L}\{Ax + Bu\}$$

$$\implies x(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}Bu(s)$$

$$y = Cx + Du \implies \mathcal{L}\{y\} = \mathcal{L}\{Cx + Du\}$$

$$\implies y(s) = \underbrace{C(sI - A)^{-1}}_{:=\Psi(s)} x(0) + \underbrace{\left(C(sI - A)^{-1}B + D\right)}_{:=G(s)} u(s)$$

Taking inverse Laplace transform, we get

$$y(t) = \Psi(t)x(0) + (G \star u)(t) = \Psi(t)x(0) + \int_0^t G(t - \tau)u(\tau)d\tau$$

$G(s)$: Transfer function

$G(t) = \mathcal{L}^{-1}\{G(s)\}$: Impulse response (Response with $u(s) = 1$)

Impulse Response of a DT LTI System

Consider the discrete time LTI System

$$x_{k+1} = Ax_k + Bu_k \implies \mathcal{Z}\{x_{k+1}\} = \mathcal{Z}\{Ax_k + Bu_k\}$$

$$\implies x(z) = (zI - A)^{-1}x_0 + (zI - A)^{-1}Bu(z)$$

$$y_k = Cx_k + Du_k \implies \mathcal{Z}\{y_k\} = \mathcal{Z}\{Cx_k + Du_k\}$$

$$\implies y(z) = \underbrace{C(zI - A)^{-1}}_{:=\Psi(z)} x_0 + \underbrace{\left(C(zI - A)^{-1}B + D\right)}_{:=G(z)} u(z)$$

$G(z)$: Transfer function

$G(t) = \mathcal{Z}^{-1}\{G(z)\}$: Impulse response (Response with $u(z) = 1$)

Realization Theory

Realization of LTI System

Given a transfer function $G(s)$, the CT state space system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

is a **realization** of $G(s)$ if $G(s) = C(sI - A)^{-1}B + D$.

For discrete time systems, replace s by z.

Zero-State Equivalence

- Many systems may realise the same transfer function
- Two state-space systems are said to be **zero-state equivalent** if they realise the same transfer function
  - They exhibit the same forced response to every input

Equivalent State-Space Systems

Algebraic Equivalence

Two CT or DT LTI systems $(A, B, C, D), (\bar{A}, \bar{B}, \bar{C}, \bar{D})$ are called **algebraically equivalent** if $\exists T, \mathtt{det}(T) \neq 0$ such that

$$\bar{A} = TAT^{-1},\ \bar{B} = TB,\ \bar{C} = CT^{-1},\ \bar{D} = D$$

The map $\bar{x} = Tx$ is called **similarity** or **equivalence** transformation.

Properties:

- With every input signal u, both systems associate the same set of outputs y (However, not the same output for same initial conditions)
- The systems are zero-state equivalent
- Algebraic Equivalence $\underset{\not\Leftarrow}{\Rightarrow}$ Zero-state Equivalence

Solutions to Homogenous LTV Systems

Given $x(t_0) = x_0 \in \mathbb{R}^n$, consider the homogenous CT LTV system

$$\dot{x}(t) = A(t)x(t), \quad t \geq 0.$$

Then, the unique solution to the above system is given by

$$x(t) = \Phi(t, t_0)x_0, \quad x(t_0) = x_0 \in \mathbb{R}^n, t \geq 0$$

$$\Phi(t, s) = I + \int_s^t A(\sigma_1)d\sigma_1 + \int_s^t A(\sigma_1) \int_s^{\sigma_1} A(\sigma_2)d\sigma_1 d\sigma_2 + \dots$$

- $\Phi_A(t, t_0) \in \mathbb{R}^{n \times n}$ - **state transition matrix** (subscript A often dropped for brevity)
- The **Peano-Baker series** defining $\Phi(t, s)$ converges for arbitrary t, s

$$A(t) = A \implies \Phi(t, s) = I + (t - s)A + \frac{(t - s)^2}{2} A^2 + \dots = e^{A(t-s)}$$

Solutions to Non-homogenous LTV Systems

Given $x(t_0) = x_0 \in \mathbb{R}^n$, consider the non-homogenous CT LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t), \quad t \geq 0.$$

Then, the unique solution to the above system $\forall t \geq 0$ is given by

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

$$y(t) = \underbrace{C(t)\Phi(t, t_0)x_0}_{:=y_h(t)} + \underbrace{\int_{t_0}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)}_{:=y_f(t)}$$

$y_h(t)$ Homogeneous (zero-input) response

$y_f(t)$ Forced (zero-initial condition) response

Solutions to Discrete LTV Systems

Given $x(t_0) = x_0 \in \mathbb{R}^n$, the unique solution to the **homogenous** DT LTV system $\forall t \in \mathbb{N}, t \geq t_0$ is given by

$$x(t+1) = A(t)x(t) \quad \implies x(t) = \Phi(t, t_0)x_0,$$

$$\Phi(t, t_0) = \begin{cases} I & t = t_0 \\ A(t-1)A(t-2)\dots A(t_0+1)A(t_0) & t > t_0 \end{cases}$$

The unique solution to the **non-homogenous** DT LTV system is

$$x(t+1) = A(t)x(t) + B(t)u(t) \quad y(t) = C(t)x(t) + D(t)u(t),$$

$$\implies x(t) = \Phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} \Phi(t, \tau+1)B(\tau)u(\tau)d\tau$$

$$y(t) = C(t)\Phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} C(t)\Phi(t, \tau+1)B(\tau)u(\tau)d\tau + D(t)u(t)$$

Matrix Differential Equation

Given $A(t) \in \mathbb{R}^{n \times n}$, let matrix differential equation in $X(t) \in \mathbb{R}^{n \times n}$ be

$$\frac{d}{dt} X(t) = A(t) X(t), \quad X(t_0) = X_0.$$

The unique continuously differentiable solution is

$$X(t) = \Phi_A(t, t_0) X_0.$$

If $X(t_0) = X_0 = I$, then $X(t) = \Phi_A(t, t_0)$.

Exercise

Show that solution of the following adjoint system (more on this later)

$$\frac{d}{dt} Z(t) = -A^\top(t) Z(t), \quad Z(t_0) = Z_0.$$

is $Z(t) = \Phi_A^\top(t_0, t) Z_0$.

Properties of State Transition Matrix

$\Phi(t, t_0)$ for CT LTV Systems

- $\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0), \quad \frac{d}{dt_0}\Phi(t, t_0) = -\Phi(t, t_0)A(t_0) \quad t \geq t_0$
- $\Phi(t, t) = I$
- For $t > s > \tau$, $\Phi(t, s)\Phi(s, \tau) = \Phi(t, \tau)$ **"Semigroup Property"**
- $\Phi(t, s)^{-1} = \Phi(s, t) \implies \Phi(t, s)$ is **non-singular**

$\Phi(t, t_0)$ for DT LTV Systems

- $\Phi(t+1, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t, t_0 - 1) = \Phi(t, t_0)A(t_0 - 1) \quad t > t_0$
- $\Phi(t_0, t_0) = I$
- For $t \geq s \geq \tau$, $\Phi(t, s)\Phi(s, \tau) = \Phi(t, \tau)$ **"Semigroup Property"**
- $\Phi(t, s)$ may be **singular**!

Solutions to CT LTI Systems - Matrix Exponential

Given $x(t_0) = x_0 \in \mathbb{R}^n$, consider the **homogenous** CT LTI system

$$\dot{x} = Ax, \quad t \geq 0.$$

Then, the unique solution to the above system is given by

$$x(t) = \Phi(t, t_0)x_0, \quad t \geq 0$$

$$\Phi(t, t_0) = \sum_{k=0}^{\infty} \frac{(t - t_0)^k}{k!} A^k = e^{A(t-t_0)}$$

$$\implies x(t) = e^{A(t-t_0)}x_0, \quad t \geq 0$$

Common Mistake:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \not\Rightarrow e^{At} = \begin{bmatrix} e^{1t} & e^{2t} \\ e^{3t} & e^{4t} \end{bmatrix}.$$

Solutions to CT LTI Systems - Matrix Exponential

Given $x(t_0) = x_0 \in \mathbb{R}^n$, consider the **non-homogenous** CT LTI system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad t \geq 0.$$

Then, the unique solution $\forall t \geq 0$ to the above system is given by

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}B(\tau)u(\tau)d\tau$$

$$y(t) = \underbrace{C(t)e^{A(t-t_0)}x_0}_{:=y_h(t)} + \underbrace{\int_{t_0}^{t} C(t)e^{A(t-\tau)}B(\tau)u(\tau)d\tau + D(t)u(t)}_{:=y_f(t)}$$

Life is easy when A is diagonal

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{1t} & 0 \\ 0 & e^{4t} \end{bmatrix}.$$

Solutions to DT LTI Systems - Matrix Exponential

Given $x(t_0) = x_0 \in \mathbb{R}^n$, consider the **homogenous** DT LTI system

$$x^+ = Ax, \quad t \in \mathbb{N}.$$

Then, the unique solution to the above system is given by

$$x(t) = \Phi(t, t_0)x_0 = A^{(t-t_0)}x_0, \quad t \in \mathbb{N}$$

The !solution with $x(t_0) = x_0 \in \mathbb{R}^n$, for **non-homogenous** DT LTI sys

$$x^+ = Ax + Bu, \quad y = Cx + Du, \quad t \in \mathbb{N}$$

$$\implies x(t) = A^{(t-t_0)}x_0 + \sum_{\tau=t_0}^{t-1} A^{t-1-\tau}Bu(\tau)$$

$$\implies y(t) = CA^{(t-t_0)}x_0 + \sum_{\tau=t_0}^{t-1} CA^{t-1-\tau}Bu(\tau) + Du(t)$$

Properties of Matrix Exponential

- $\frac{d}{dt}e^{At} = Ae^{At}, \quad e^{A\cdot 0} = I, \quad t \geq 0$
- Generally, $e^{(A+B)t} \neq e^{At}e^{Bt}$.
- If $AB = BA \implies e^{(A+B)t} = e^{At}e^{Bt}$
- $e^{At}e^{A\tau} = e^{A(t+\tau)}, \quad \forall t, \tau \in \mathbb{R}$ **Semigroup Property**
- $\left(e^{At}\right)^{-1} = e^{-At}$
- $Ae^{At} = e^{At}A, \quad \forall t \in \mathbb{R}$
- Due to Cayley-Hamilton theorem, we see that

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t)A^i, \quad \forall t \in \mathbb{R}$$

- $\mathtt{det}(e^{At}) = e^{\mathtt{Tr}(A)t}$

Computing Matrix Exponential

Continuous Time Case - Use Laplace transform

$$\frac{d}{dt}e^{At} = Ae^{At} \implies \mathcal{L}\left\{\frac{d}{dt}e^{At}\right\} = \mathcal{L}\left\{Ae^{At}\right\}$$

$$\implies e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

Hurwitz: $\texttt{Real(eig}(A)) < 0 \implies \lim_{t\to\infty} e^{At} \to 0_{n\times n} \implies y(t) \to y_f(t)$

Discrete Time Case - Use Z transform

$$\mathcal{Z}\left\{A^{t+1}\right\} = z\left(\mathcal{Z}\left\{A^t\right\} - I\right) \implies A^t = \mathcal{Z}^{-1}\left[z(zI - A)^{-1}\right]$$

Schur: $|\texttt{eig}(A)| < 1 \implies \lim_{t\to\infty} A^t \to 0_{n\times n} \implies y(t) \to y_f(t)$

Stability of LTV Systems

Recall Stability of LTI Systems

Hurwitz: $\texttt{Real(eig}(A)) < 0 \implies \lim_{t\to\infty} e^{At} \to 0_{n\times n} \implies y(t) \to y_f(t)$

Schur: $|\texttt{eig}(A)| < 1 \implies \lim_{t\to\infty} A^t \to 0_{n\times n} \implies y(t) \to y_f(t)$

Note: While calculating e^{At} for CT LTI systems, remember to use the Jordan form when eigenvalues have geometric multiplicity ≥ 2.

Stability of CT and DT LTV Systems

- Stability for a CT time-varying system $\dot{x}(t) = A(t)x(t)$ cannot be determined by the eigenvalues of $A(t)$
- Location of the eigenvalues $\lambda(A(t))$ in LHP $\forall t \geq 0$ is neither sufficient nor necessary condition for stability (more on this topic later)

State Transition Matrices With Variable Change (CT)

Using the variable change $x(t) = P(t)z(t)$ (with $P(t)$ invertible),

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = P(t_0)z(t_0) = x_0 \iff$$

$$\dot{z}(t) = [P(t)^{-1}A(t)P(t) - P(t)^{-1}\dot{P}(t)]z(t), \quad z(t_0) = P(t)^{-1}x_0$$

$$\implies \Phi_{P^{-1}AP-P^{-1}\dot{P}}(t,t_0) = P(t)^{-1}\Phi_A(t,t_0)P(t_0)$$

Proof:

$$APz = Ax = \dot{x} = \frac{d}{dt}(Pz) = \dot{P}z + P\dot{z} \implies P\dot{z} = [AP - \dot{P}]z$$

$$\implies \dot{z} = [P^{-1}AP - P^{-1}\dot{P}]z \implies z(t) = \Phi_{P^{-1}AP-P^{-1}\dot{P}}(t,t_0)z(t_0)$$

$$z(t) = P(t)^{-1}x(t) = P(t)^{-1}\Phi_A(t,t_0)x(t_0)$$

$$= \underbrace{P(t)^{-1}\Phi_A(t,t_0)P(t_0)}_{\Phi_{P^{-1}AP-P^{-1}\dot{P}}(t,t_0)} z(t_0)$$

State Transition Matrices With Variable Change (DT)

Using the variable change $x(k) = P(k)z(k)$ (with $P(k)$ invertible),

$$x(k+1) = A(k)x(k), \quad x(k_0) = P(k_0)z(k_0) = x_0 \iff$$

$$z(k+1) = [P(k+1)^{-1}A(k)P(k)]z(k)$$

$$\implies \Phi_z(k,j) = P(k)^{-1}\Phi_x(k,j)P(j)$$

Proof: Very similar to the continuous case (use Z transform instead of Laplace transform)

Abel-Jacobi-Liouville Theorem

Volume Interpretation: The determinant of a matrix A is the oriented volume of the parallelepiped P whose edges are given by columns of A.

$$|\mathtt{det}(A)| = \mathtt{vol}(P)$$

Abel-Jacobi-Liouville Theorem: Exercise: Prove it.

Let $A(t)$ be continuous. Then,

$$\mathtt{det}(\Phi(t,\tau)) = e^{\int_\tau^t \mathtt{Tr}[A(\sigma)]d\sigma}$$

$$\frac{d}{dt}\mathtt{det}(\Phi(t,\tau)) = \mathtt{Tr}[A(t)]\,\mathtt{det}(\Phi(t,\tau))$$

- Interpretation: Volume contracts. (Recall: $\mathtt{Tr}(A) = \sum \mathtt{eig}(A)$)
- $\mathtt{det}(\Phi(t,\tau)) > 1 (< 1, = 1) \implies$ vol. expands (shrinks, stays const.)

Example

Question

Check if the following oscillative system can be made asymptotically stable through the use of an output feedback $u(t) = -k(t)y(t)$.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).$$

Solution:

Apply Abel-Jacobi-Liouville theorem to the closed loop system matrix

$$A_c = \begin{bmatrix} 0 & 1 \\ -(1+k(t)) & 0 \end{bmatrix} \implies \mathtt{det}(\Phi(t,0)) = e^{t\mathtt{Tr}(A_c)} = 1.$$

System cannot be made asymptotically stable, as $\Phi(t,0) \nrightarrow 0$ as $t \to \infty$.

Linear Time Periodic (LTP) Systems

Definition of Periodic Matrices

A matrix $A \in \mathbb{R}^{n \times n}$ is called **T-periodic** if $\exists T > 0$ such that $\forall t \geq 0$

$$A(t+T) = A(t)$$

The smallest T for which above equation holds true is called the **period**.

CT Linear Time Periodic (LTP) Systems

A state space system $\Sigma : (A, B, C, D)$ is called **T-periodic** if all matrices (A, B, C, D) are **T-periodic**. For eg.

$$\dot{x}(t) = A(t)x(t) \quad \text{is } \textbf{T-periodic} \text{ LTP system } \iff \quad A(t+T) = A(t)$$

DT (LTP): $x(k+1) = A(k)x(k)$ is K-periodic $\iff A(k+K) = A(k)$

Floquent Decomposition for LTP Systems

CT Floquent Decomposition

For a given **T-periodic** matrix $A \in \mathbb{R}^{n \times n}$, its transition matrix is

$$\Phi(t,\tau) = P(t)\, e^{R(t-\tau)}\, P^{-1}(\tau), \tag{1}$$

$R \in \mathbb{R}^{n \times n}$ constant (even complex) - Average of $A(t)$ over 1 period. R is selected such that $e^{RT} = \Phi(T,0)$.

$P(t) \in \mathbb{R}^{n \times n}$ is differentiable, invertible & T-periodic. $P(t)$ is selected such that $P(t) = \Phi(t,0)e^{-Rt}$.

$\Longrightarrow \Phi(t,\tau)$ is $\Phi(t,\tau) = \Phi(t,0)\Phi(0,\tau) = P(t)\, e^{R(t-\tau)}\, P^{-1}(\tau)$

Eg., $A(t) = \begin{bmatrix} -1 & 0 \\ -\cos(t) & 0 \end{bmatrix} \Longrightarrow R = \begin{bmatrix} -1 & 0 \\ -\frac{1}{2} & 0 \end{bmatrix}, P(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}(\cos(t) - \sin(t) - 1) & 1 \end{bmatrix}$

Properties of LTP Systems

- Let $A(t) \in \mathbb{R}^{n \times n}$ be T-periodic. Then, $\forall t_0 \geq 0, \exists x_0 \neq 0$ such that solution of $\dot{x}(t) = A(t)x(t)$ for $x(t_0) = x_0$ is **T-periodic** iff $\exists \lambda(e^{RT}) = 1$.
- A solution $x(t)$ of a T-periodic system $\dot{x}(t) = A(t)x(t) + f(t)$ with T-periodic matrices $A(t), f(t)$ is **T-periodic** iff $x(t_0 + T) = x(t_0)$.
- For a CT LTI system, $\dot{x}(t) = Ax(t) + Bu(t)$ with $x(0) = x_0$, let $\nexists \mathtt{Real}(\lambda(A)) = 0$. Then, $\forall$ T-periodic input $u(t)$, $\exists x_0$ such that solution $x(t)$ is T-periodic and unique.
- For DT LTP systems, $\Phi(k, j) = P(k)\, R^{(k-j)}\, P^{-1}(j)$