LUNDS TEKNISKA HÖGSKOLA
MATEMATISKA INSTITUTIONEN

TENTAMENSSKRIVNING
Matristeori
December 7–17, 2020

Hand in solutions of 6 from 8 problems below (7 from 8 problems for NF students). For a passing grade (3 for LTH students and graduate students, 3.5 for NF students), at least one problem of the last four have to be solved. Credits can be given for partially solved problems. Write your solutions neatly and explain your calculations. Both the content and the format of your solutions, and also how difficult problems you choose, will affect your grade.

The exam has to be sent by e-mail to victor.ufnarovski@math.lth.se at the latest December 17. **Write your name, section-year (or subject for Ph.D-students), id-number, phone number and email address on the first page, and write your name on each of the following pages. Write also preferable times for the oral part.** Make sure that you get a confirmation that your mail was received! Write explicitly if you are student (LTH or NF) or graduate student. The result will be sent to the email address written on the first page. The oral part of the exam should take place in December-January (depending on your schedule).

You may use any books and computer programs (e.g. Matlab and Maple), but it is not permitted to get help from other persons, including from internet. Programs and long calculations can be submitted by e-mail separately. You do not need to explain the elementary matrix operation such as matrix multiplications made by the computer, but you should explain steps in more complicated calculations such as jordanization. In such cases you can control your results by computer but do not need to submit such calculations. Ask if you are not sure!

You can use any theorem in the book without proof but not exercises without explaining the solutions.

All matrices below are complex matrices if nothing else is specified explicitly.

Good luck!

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1. Let A be a non-zero square matrix such that $A^2 = A$. Show that A is diagonalizable and that for any operator norm we have $\|A\| \geq 1$.

2. Let $A = \sum_{i=1}^{n-1} i \cdot E_{i,i+1}$ be $n \times n$ matrix and $B = e^A$. Show that $b_{ij} = \binom{j-1}{i-1}$ for $i \leq j$.

3. Let A and B be two matrices of size 2×2 such that $AB = BA$. Prove that either they are proportional or at least one of them has a trace different from zero.

4. Let A be a square matrix.

a) Show A can be written as a sum $A = A_1 + A_2$ where A_1 is Hermitian and A_2 is skew-Hermitian .

b) Show that the decomposition is unique .

c) Show that A is normal if and only if A_1 and A_2 commute.

5. Let A and B be two square matrices of size $n \leq 3$ with the same trace. Suppose that for any polynomial $p(x)$ we have $p(A) = 0 \Leftrightarrow p(B) = 0$. Show that A and B are similar. Is the same statement true for any size n?

6. Prove that a square matrix A is normal if and only if A commutes with $A^H A$.

7. Let $A = (a_{ij})$ be a Hermitian $n \times n$ matrix written in block form as

$$A = \begin{pmatrix} a_{11} & X^H \\ X & B \end{pmatrix}$$

where X is a column of size $n-1$ and B is a matrix of size $n-1 \times n-1$. Show that

$$\det A = a_{11} \det B - X^H (\operatorname{adj} B) X,$$

where the product containing the adjugate matrix is interpreted as a number.

8. Let A be a square matrix. Peter suggests the following algorithm to find the inverse matrix A^{-1}. He starts from a block matrix

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$$

and using row and column operations transfer this matrix to

$$\begin{pmatrix} B & I \\ 0 & C \end{pmatrix}$$

He claims that $CB = A^{-1}$. He illustrates his algorithm with the following example:

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ -3 & 1 & 0 & -2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -3 & 1 & 0 & -2 \\ \hline 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -3 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \frac{-1}{2} \end{array}\right).$$

a) Check that here $CB = A^{-1}$. Is this always true?

b) He claims also that if A is a positive definite Hermitian matrix then he can get B to be lower triangular and C to be upper triangular. He also claims that his decomposition in this case gives a Cholesky decomposition of A. Which of those statements are true?