LUNDS TEKNISKA HÖGSKOLA
MATEMATISKA INSTITUTIONEN

TENTAMENSSKRIVNING
Matristeori
January 4–14, 2021

Hand in solutions of 6 from 8 problems below (7 from 8 problems for NF students). For a passing grade (3 for LTH students and graduate students, 3.5 for NF students), at least one problem of the last four have to be solved. Credits can be given for partially solved problems. Write your solutions neatly and explain your calculations. Both the content and the format of your solutions, and also how difficult problems you choose, will affect your grade.

The exam has to be sent by e-mail to victor.ufnarovski@math.lth.se at the latest January 14. **Write your name, section-year (or subject for Ph.D-students), id-number, phone number and email address on the first page, and write your name on each of the following pages. Write also preferable times for the oral part.** Make sure that you get a confirmation that your mail was received! Write explicitly if you are student (LTH or NF) or graduate student. The result will be sent to the email address written on the first page. The oral part of the exam should take place in December-January (depending on your schedule).

You may use any books and computer programs (e.g. Matlab and Maple), but it is not permitted to get help from other persons, including from internet. Programs and long calculations can be submitted by e-mail separately. You do not need to explain the elementary matrix operation such as matrix multiplications made by the computer, but you should explain steps in more complicated calculations such as jordanization. In such cases you can control your results by computer but do not need to submit such calculations. Ask if you are not sure!

You can use any theorem in the book without proof but not exercises without explaining the solutions.

All matrices below are complex matrices if nothing else is specified explicitly.

Good luck!

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1. Find the singular values and SV-decomposition of the matrix

$$A = \begin{pmatrix} i & 1 \\ i & -1 \\ -i & -1 \end{pmatrix}.$$

For this A and $B = (1\ \ 2\ \ 2)^T$, find 2 x 1 matrix X that minimizes $\|AX - B\|_2$.

2. Let A be a 5×5 matrix such that $a_{ij} = 2021$ for each i, j. Find $\sin A, \cos A, \sin 2A$ and show that $\sin 2A = 2 \sin A \cos A$. Is this true for any square matrix A?

3. Let $k \geq 1$. Show that a non-zero matrix has rank $r \leq k$ if and only if it can be written as a sum of k matrices of rank 1.

4. Show that if four matrices form a basis for the set of all 2×2 matrices then at least one of them is not nilpotent.

5. Let A be a square matrix such that $\mathbf{rank}\, A = 2$ and $\mathbf{tr}\, A = 0$. Show that either A is diagonalizable or $A^3 = 0$.

6. Let A be a square matrices and $p_A(x) = \prod_{i=1}^{l}(x - \lambda_i)^{m_i}$, $\pi_A(x) = \prod_{i=1}^{l}(x - \lambda_i)^{s_i}$ be its characteristic and minimal polynomial. Let $B = e^A$. Which of the following statements are true?

1. $\prod_{i=1}^{l}(x - e^{\lambda_i})^{m_i}$ is the characteristic polynomial of the matrix B.
2. $\prod_{i=1}^{l}(x - e^{\lambda_i})^{s_i}$ is the minimal polynomial of the matrix B.

Motivate your answer with a proof or a counterexample.

7. Prove that a square matrix A is Hermitian if and only if $A^H A = A^2$.

8. Let $A = (a_{ij})$ be $n \times n$ matrix consisting of binomial coefficients, $a_{ij} = \binom{i-1+j-1}{i-1}$. Prove that this matrix is positive definite.