

## **FRTF15 Control Theory Assignment 2**

**To be handed in Sunday 21th of March 2021, 23:59, at the latest**

### **Information About the Hand-in Assignments**

The course FRTF15 Control Theory contains two mandatory hand-in assignments. The exercises range from investigations of theoretical concepts to simulation exercises using numerical tools, such as MATLAB.

The intention of the exercises is that you should apply your knowledge from complex and linear analysis in order to solve the tasks given. Some of the exercises have more of a discussion character, and require that you actively use the available literature.

Assignment 1 is performed in a group, whereas Assignment 2 is performed individually. Assignment 2 should be submitted as a written report.

### **Report**

The written report is an important part of the work, and it is evaluated both with respect to content and layout. The report should be written such that engineering students on your own level can follow the solutions.

Apply the guidelines for writing reports that you have obtained in previous courses. Some hints:

- The report can be written with paper and pencil or using appropriate computer software. Write complete sentences.
- If possible, start by describing the problem to be solved. Also, explain the notation and variables that you introduce.
- Explain the different steps of your solution and provide logical motivations for them. Specify the theorems or results that you utilize (Pythagoras Theorem, Cauchy's Integral Formula, ...).
- Please make sure that you verify that all questions in the exercises are answered in your solutions.
- Attach figures and plots if it contributes to the interpretation of the solutions. Also attach relevant code or scripts if you employ numerical tools for simulations and computations.

- It is allowed to discuss the exercises with other students on a general level, but each student should submit an individual report.

### Grading

In the course FRTF15 Control Theory, the grades are Pass and Fail. In order to pass the course, both assignments need to be approved.

In order for Assignment 2 to be approved, the report must have almost correct solutions to all of the problems and be sufficiently well written. If the report is deemed insufficient, it will be returned and you will get 7 days to revise it based on the feedback and resubmit it for reevaluation.

### Matlab Hints

Practical hints to using MATLAB can be found in the document *A Matlab Tutorial*, available on the course homepage. Particularly useful commands are:

<b>ss</b>	Create a state-space model
<b>tf</b>	Create a transfer function
<b>bode</b>	Plot Bode diagram
<b>bodemag</b>	Plot Bode magnitude
<b>place</b>	Pole placement
<b>obsv</b>	Observability matrix
<b>ctrb</b>	Controllability matrix
<b>zero</b>	Compute zeros
<b>pole</b>	Compute poles
<b>lsim</b>	Simulate a state-space system

1. Consider a process that resembles the spring-mass-damper system investigated in Laboratory Exercise 3 in the Basic Course in Control:

$$\begin{aligned}\frac{d^2 p_1}{dt^2} &= (p_2 - p_1) - 0.1 \frac{dp_1}{dt} + u, & p(0) &= p_0 \\ \frac{d^2 p_2}{dt^2} &= (p_1 - p_2) - 0.1 \frac{dp_2}{dt}, & \dot{p}(0) &= \dot{p}_0 \\ y &= p_1\end{aligned}$$

- a. Find a state-space representation and use the command `place` in MATLAB in order to compute a state-feedback controller based on estimated states,

$$u = -L\hat{x} + l_r r,$$

which gives the following poles of the closed-loop system:

$$(s^2 + s + 3/2)(s^2 + s + 1),$$

and an observer

$$\dot{\hat{x}} = (A - KC)\hat{x} + Bu + Ky \quad \hat{x}(0) = 0,$$

with the pole placement:

$$(s^2 + s + 3/2)(s^2 + 2\zeta_o\omega_o s + \omega_o^2),$$

where  $\zeta_o = 0.5$  and  $\omega_o = 1$ . The static gain should be 1.

- b. Simulate the step response of the closed-loop system ( $r(t) = 1, t \geq 0$ )

$$\begin{aligned}\dot{x} &= Ax - BL\hat{x} + Bl_r r, & x(0) &= x_0 \\ \dot{\hat{x}} &= KCx + (A - KC - BL)\hat{x} + Bl_r r, & \hat{x}(0) &= 0,\end{aligned}$$

for different values of  $\omega_o$  (for example, 0.75, 2, 10) in the cases when  $x_0 = 0$  and  $x_0 \neq 0$ . It might be helpful to introduce the matrices

$$\begin{aligned}A_e &= [A, -B*L; K*C, A-K*C-B*L] \\ B_e &= [B*l_r; B*l_r]\end{aligned}$$

In MATLAB, the function `lsim` can be used. Alternatively, the simulation environment SIMULINK can be used for simulating the system. Explain the differences in the obtained simulation results. In particular, investigate the estimation error  $\tilde{x} = x - \hat{x}$ .

- c. Determine the transfer function of the process on the form  $Y(s) = P(s)U(s)$ . Also, compute the transfer function of the controller according to the expression

$$U(s) = -G_{uy}(s)Y(s) + G_{ur}(s)R(s).$$

The transfer functions should be expressed in the variables  $s, A, B, C, L$ , and  $K$ .

- d. The loop-transfer function for the compensated system is given by  $L(s) = P(s)G_{uy}(s)$ . Plot the Bode diagram and determine the amplitude and phase margins for the case  $\omega_o = 1$  (use `margin` in MATLAB).

Also, plot the magnitude of the sensitivity function,  $|S(i\omega)|$ , using `bodemag` and determine its maximum value  $M_s$ . What does this value of  $M_s$  mean for the robustness of the controlled system?

2. Consider the segway robot illustrated in Figure 1. The pendulum angle relative to the vertical plane is  $\theta$  and the wheel angle is  $\phi$ . The applied torque is denoted  $\tau$ .

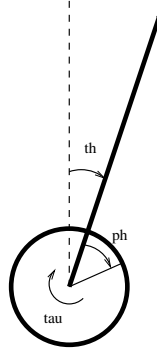


Figure 1: A schematic picture of a segway robot.

The equations of motion can then be modeled by Euler-Lagrange mechanics, which result in the model

$$\begin{pmatrix} J_p & m_p l r \cos \theta \\ m_p l r \cos \theta & J_w + m_w r^2 + m_p r^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} + \begin{pmatrix} -m_p g l \sin \theta \\ -m_p l r \dot{\theta}^2 \sin \theta \end{pmatrix} = \begin{pmatrix} -\tau \\ \tau \end{pmatrix}$$

where  $J_p$  [kgm<sup>2</sup>] is the moment of inertia of the body with respect to the pivot,  $m_p$  [kg] is the mass of the pendulum,  $l$  [m] is the distance from the pivot from the body center of mass,  $g$  [m/s<sup>2</sup>] is the gravitational acceleration, and  $m_w$  [kg],  $r$  [m],  $J_w$  [kgm<sup>2</sup>] are the mass, radius and moment of inertia of the wheel assembly, respectively.

- a. Introducing the state vector  $x = (\theta, \dot{\theta}, \phi, \dot{\phi})$  and linearizing around the origin yields a state-space equation on the form

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -c & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ -b \\ 0 \\ d \end{pmatrix} \tau.$$

Determine  $a, b, c, d$ .

- b. Determine the transfer function from  $\tau$  to  $\phi$ .
- c. Determine the zeros and poles of the system, when  $\tau$  is the input and  $\phi$  is the output.
- d. Is the system observable using the angle  $\theta$  as measurement signal? If not, determine the unobservable subspace.
- e. Determine conditions on the parameters  $a, b, c, d$  for the system *not* to be controllable. Relate the conditions to the expression for the transfer function derived in Exercise b. Which states can be reached in these cases?

- f. Determine the transfer functions in the empty boxes in Figure 2.

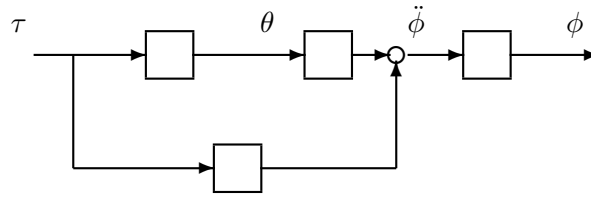


Figure 2: Block diagram for the linearized segway dynamics.

- g. Can you stabilize the upright position of the pendulum using a feedback law of the form  $\tau = -l_1\theta - l_2\dot{\theta}$ ?
- h. Let  $\theta$  be the measurement signal and use a feedback law of the same form as in Exercise g. The angular measurements are disturbed by noise in the frequency range 50–1000 rad/s. The physical parameter values are given by  $J_p = 0.16$ ,  $m_p = 2.9$ ,  $l = 0.17$ ,  $g = 9.81$ ,  $m_w = 0.5$ ,  $r = 0.05$ ,  $J_w = 0.0005$ . Suggest suitable values of the parameters  $l_1$  and  $l_2$ . In particular, plot the Bode diagram for the sensitivity and complementary sensitivity functions for your choice of parameters.

**3.****a.** Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad x(0) = 0.$$

Determine the piece-wise constant input signal  $u$ ,

$$u(t) = \begin{cases} u_1, & 0 \leq t \leq 1/2 \\ u_2, & 1/2 < t \leq 1 \end{cases}$$

that solves

$$\begin{aligned} & \underset{u}{\text{minimize}} && \int_0^1 u(t)^2 dt \\ & \text{subject to} && x(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

**b.** What happens in the case that

$$u(t) = \begin{cases} u_1, & 0 \leq t \leq 1/3 \\ u_2, & 1/3 < t \leq 2/3 \\ u_3, & 2/3 < t \leq 1 \end{cases}$$

**c.** Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad x(0) = 0,$$

and redo the computations in Exercise a. Explain the result.