LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

Control Theory
Assignment 1

To be handed in Sunday 14th of February 2021, 23:59, at the latest

Information About the Hand-in Assignments

The course FRTF15 Control Theory contains two mandatory hand-in assignments. The exercises range from investigations of theoretical concepts to simulation exercises using numerical tools, such as MATLAB.

The intention of the exercises is that you should apply your knowledge from complex and linear analysis in order to solve the tasks given. Some of the exercises have more of a discussion character, and require that you actively use the available literature.

Assignment 1 is performed in groups of three or four students. Everyone in the group should produce an individual set of solutions according to the following instructions:

- Although the tasks are solved together within the group, each student should hand in a complete set of solutions on their own.
- The hand-in can be written with either paper and pencil or using appropriate computer software. Write complete sentences.
- Explain the different steps of your solution and provide logical motivations for them. Specify the theorems or results that you utilize (Pythagoras Theorem, Cauchy's Integral Formula, . . .).
- Attach figures and plots if it contributes to the interpretation of the solutions. Also attach relevant code or scripts if you employ numerical tools for simulations and computations.

The solutions should then be presented orally at a seminar together with other groups. At this session, groups are also expected to discuss each other's solutions.

Grading

In the course FRTF15 Control Theory, the grades are Pass and Fail. In order to pass the course, both hand-in assignment reports need to be approved. For the first hand-in, an acceptable presentation is also necessary.

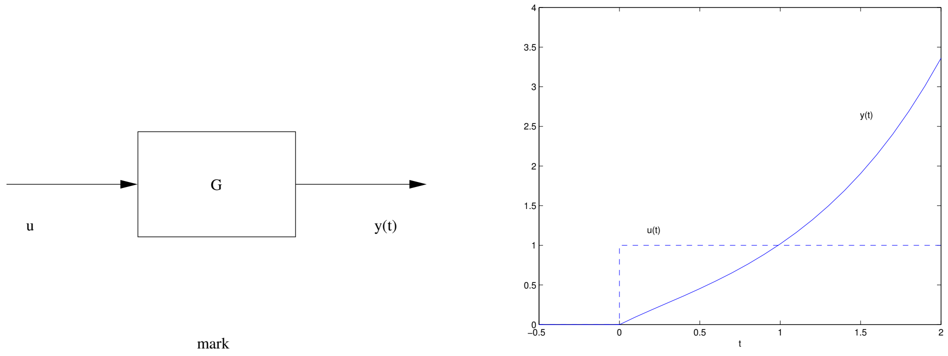

Figure 1: The system and its step response in Exercise 1a).

1.

a. In an experiment a step function $u(t) = 1$, $t \geq 0$, is applied to a linear time-invariant system $\mathcal{G}$ at rest ($x(0) = 0$). The step response is shown in Figure 1, which is described by the function

$$y(t) = \frac{1}{2}\left(e^{t} - \frac{1}{3}e^{-3t} - \frac{2}{3}\right), \quad t \geq 0.$$

Calculate the impulse response.

b. Compute the transfer function $G(s)$ for the system $\mathcal{G}$ in exercise a) by using the definition of the Laplace transform. The Laplace transform $G(s)$ only converges in a half-plane $\operatorname{Re} s > \alpha \in \mathbb{R}$. Determine α.

c. Show that the general solution $y(t)$ for a linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \ , \ \ x(0) = x_0 \\ y &= Cx + Du \end{aligned}$$

satisfies

$$y(t) = |G(i\omega)| \sin\left(\omega t + \arg(G(i\omega))\right) + Ce^{At}\left(x_0 - \bar{x}_0(\omega)\right),$$

when $u(t) = \sin \omega t = (e^{i\omega t} - e^{-i\omega t})/2i$, $t \geq 0$, where ω is a real number such that $i\omega$ is not an eigenvalue of A. In particular, find expressions for the frequency response $G(i\omega)$ and the part of the transient denoted by $\bar{x}_0(\omega)$.

d. The linearized dynamics for the system with two water tanks which is used in laboration 1 and 2 in the basic course FRT010 Reglerteknik AK can be described by

$$\begin{aligned} \dot{x}_1 &= -\frac{\gamma_1}{2\sqrt{x_1^0}}x_1 + \delta u \\ \dot{x}_2 &= \frac{\gamma_1}{2\sqrt{x_1^0}}x_1 - \frac{\gamma_2}{2\sqrt{x_2^0}}x_2 \end{aligned}$$

where x_i is the height of tank i. Choose stationary point ($x_1^0 = 0.5, x_2^0 = 0.5$) and parameter values according to the manual for laboration 2. Simulate the

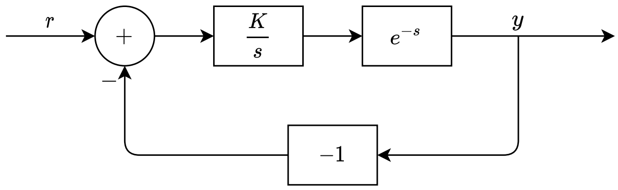

Figure 2: The system in Exercise 2a).

linearized model with the input signal $u(t) = A\sin(\omega t)$ using MATLAB for various amplitudes A, frequencies ω, and initial conditions x_0. In particular, investigate what happens when the initial state is chosen so that the transient response vanishes, by using the result of exercise c).

Hint: The function `lsim` in MATLAB can be used to simulate systems on state-space form created by the function `ss`.

2.

a. An example of a system whose transfer functions is not rational is the system in Figure 2. For which $K > 0$ is this (closed-loop) system asymptotically stable?

Sketch the output signal in the time interval $t \in [0, 3]$ when a step $r(t) = 1$ is applied to the closed-loop system at $t = 0$ when the system is at rest.

b. In order to come up with solutions that work in real life, and not only in theory, it is a good idea to ensure that your solution is applicable to not only your problem at hand, but rather for a class of similar problems. In this exercise we will study what make and what does not make systems similar from a feedback perspective. A quite natural method would be to compare step responses of open systems. Explain what could go wrong under the assumption that systems with similar step responses are "close enough" to a nominal system to be controlled with the same controller. In particular, you may examine the below systems P_1, P_2 and P_3, with step responses in Fig. 3 ($k = 100,\ T = 0.025$),

$$\begin{aligned} P_1(s) &= \frac{k}{1+s} \\ P_2(s) &= \frac{k}{-1+s} \\ P_3(s) &= \frac{k}{1+s}\left(\frac{1-s/T}{1+s/T}\right). \end{aligned}$$

c. Consider the systems

$$\begin{aligned} P_1(s) &= \frac{10}{(1+10s)(1+0.05s)}, \\ P_2(s) &= \frac{10}{(1+10s)(1-0.05s)}, \end{aligned}$$

with nyquist curves depicted in Fig. 4. Notice the similarity of their frequency responses. Discuss what happens when you close the loops of these two systems using unit feedback. What must we take into consideration when comparing frequency reponses?

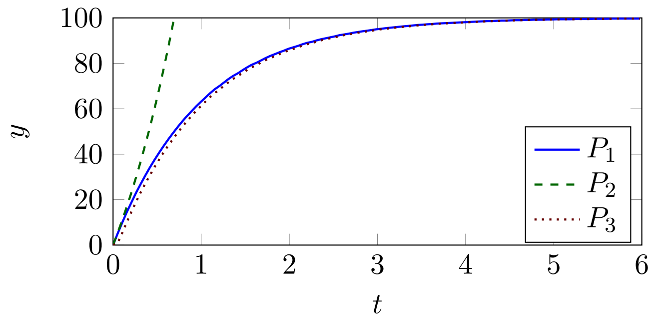

Figure 3: Step responses of the systems in (b).

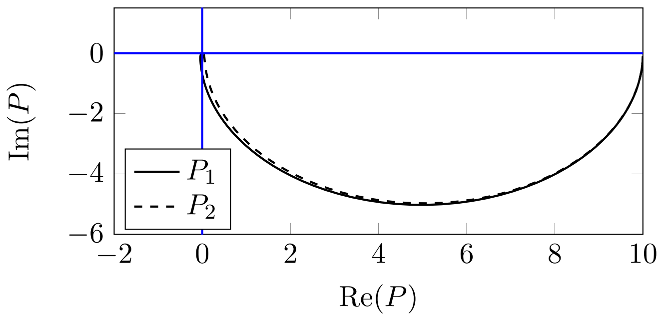

Figure 4: Nyquist response of the systems in (c).

d. Use the Nyquist criterion to find the values of $K > 0$ such that the closed-loop system obtained by closing

$$L(s) = K\frac{2s+1}{s^2(s+1)(0.4s+1)}$$

in negative feedback is asymptotically stable. In particular, discuss the image of the small semi-circle. Code that might be useful:

```
b=[0,0,0,2,1]; a=[0.4,1.4,1,0,0]; L=tf(b,a)
fi=[-1:0.01:1]*pi/2; circ=exp(i*fi);
wmin=0.5;wmax=10;
bs=polyval(b,wmin*circ);
as=polyval(a,wmin*circ);
gcirc=bs./as;
plot(real(gcirc),imag(gcirc))
hold on
nyquist(L,{wmin,wmax})
```

The Nyquist criterion also describes how the *number* of poles in the right half plane varies for different values of $K > 0$. Use MATLAB to verify your result by computing the roots of the closed-loop denominator polynomial for interesting values of K. Code that might be useful:

```
b=[0,0,0,2,1]; a=[0.4,1.4,1,0,0];
roots(a+K*b)
```

What happens if the open-loop system instead is

$$L(s) = K\frac{2s-1}{s^2(s+1)(0.4s+1)}?$$

3.

a. A version of Bode's relations that is useful for numerically computing the phase given the gain is

$$\arg G(i\omega_0) = \frac{2\omega_0}{\pi} \int_0^\infty \frac{\log|G(i\omega)| - \log|G(i\omega_0)|}{\omega^2 - \omega_0^2} d\omega,$$

where log refers to the natural logarithm. Use this formula to compute the phase of the transfer function

$$G_1(s) = \frac{1}{s+1}.$$

In MATLAB numerical integration can be done using the function `integral`. What happens for

$$G_2(s) = \frac{1}{s-1}?$$

Compare your results with the corresponding values obtained with the analytical expression for the phase.

b. Consider the Bode diagram for

$$G(s) = \frac{(s+1/2)(s+2)}{s(s+1/4)(s+1)(s+4)}.$$

Find a $\hat{G}(s)$ whose gain $|\hat{G}(i\omega)|$ is a straight line in a loglog diagram and which approximates $|G(i\omega)|$ between $\omega = 1/4$ and $\omega = 4$. That is, find K and a in

$$\hat{G}(s) = \frac{K}{s^a},$$

where the slope a does not have to be an integer. Compute the phase $\arg G(i)$ and compare with $\arg \hat{G}(i)$. Discuss the connection between the slope of the line and the phase. Code that might be useful:

```
om=logspace(log10(1/4),log10(4));
F_mag=K./om.^a;
loglog(om,F_mag)
```

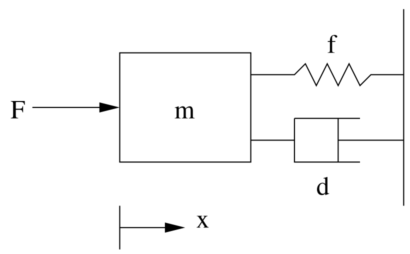

Figure 5: The system in Exercise 4.

4. The mechanical system in Figure 5 with a mass, spring, and damper is described by

$$m\ddot{x} = -d\dot{x} - fx + F$$

a. Find the transfer function from F to x.

b. Let the system be controlled by a proportional controller with gain K. Compute the sensitivity function $S(s)$ when $m = f = d = 1$. What are the different interpretations of $S(s)$?

c. Use MATLAB to plot the Bode diagram of $S(s)$ for the three cases $K = 1, 5, 10$. Discuss the relation of the plots to Bode's formula.