formulae in Theorems 16.4 and 16.5. This approach is exactly the procedure used in Doyle [1984] and Francis [1987] with Lemma 15.7 used to solve the general distance problem. Although this approach is conceptually straightforward and was, in fact, used to obtain the first proof of the current state space results in this chapter, it seems unnecessarily cumbersome and indirect.

17.8 State Feedback and Differential Game

It has been shown in Chapters 15 and 16 that a (central) suboptimal full information $\mathcal{H}_\infty$ control law is actually a pure state feedback if $D_{11} = 0$. However, this is not true in general if $D_{11} \neq 0$, as will be shown below. Nevertheless, the state feedback $\mathcal{H}_\infty$ control is itself a very interesting problem and deserves special treatment. This section and the section to follow are devoted to the study of this state feedback $\mathcal{H}_\infty$ control problem and its connections with full information control and differential game.

Consider a dynamical system

$$\dot{x} = Ax + B_1 w + B_2 u \tag{17.7}$$
$$z = C_1 x + D_{11} w + D_{12} u \tag{17.8}$$

where $z(t) \in \mathbb{R}^{p_1}$, $y(t) \in \mathbb{R}^{p_2}$, $w(t) \in \mathbb{R}^{m_1}$, $u(t) \in \mathbb{R}^{m_2}$, and $x(t) \in \mathbb{R}^n$. The following assumptions are made:

(AS1) (A, B_2) is stabilizable;

(AS2) There is a matrix $D_\perp$ such that $\begin{bmatrix} D_{12} & D_\perp \end{bmatrix}$ is unitary;

(AS3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.

We are interested in the following two related quadratic min-max problems: given $\gamma > 0$, check if

$$\sup_{w \in B\mathcal{L}_2[0,\infty)} \min_{u \in \mathcal{L}_2[0,\infty)} \|z\|_2 < \gamma$$

and

$$\min_{u \in \mathcal{L}_2[0,\infty)} \sup_{w \in B\mathcal{L}_2[0,\infty)} \|z\|_2 < \gamma.$$

The first problem can be regarded as a full information control problem since the control signal u can be a function of the disturbance w and the system state x. On the other hand, the optimal control signal in the second problem cannot depend on the disturbance w (the worst disturbance w can be a function of u and x). In fact, it will be shown that the control signal in the latter problem depends only on the system state; hence, it is equivalent to a state feedback control.

Theorem 17.6 *Let $\gamma > 0$ be given and define*

$$R := D_{1\bullet}^* D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \textit{where} \quad D_{1\bullet} := [D_{11} \;\; D_{12}]$$

$$H_\infty := \begin{bmatrix} A & 0 \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C_1^* D_{1\bullet} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\bullet}^* C_1 & B^* \end{bmatrix}$$

where $B := \begin{bmatrix} B_1 & B_2 \end{bmatrix}$.

(a) $\sup_{w \in B\mathcal{L}_2[0,\infty)} \min_{u \in \mathcal{L}_2[0,\infty)} \|z\|_2 < \gamma$ *if and only if*

$$\overline{\sigma}(D_\perp^* D_{11}) < \gamma, \quad H_\infty \in dom(Ric), \quad X_\infty = Ric(H_\infty) \geq 0.$$

Moreover, an optimal u is given by

$$u = -D_{12}^* D_{11} w + \begin{bmatrix} D_{12}^* D_{11} & I \end{bmatrix} Fx,$$

and a worst w_{fiworst} is given by

$$w_{\text{fiworst}} = F_{1\infty} x$$

where

$$F := \begin{bmatrix} F_{1\infty} \\ F_{2\infty} \end{bmatrix} := -R^{-1} \left[D_{1\bullet}^* C_1 + B^* X_\infty \right].$$

(b) $\min_{u \in \mathcal{L}_2[0,\infty)} \sup_{w \in B\mathcal{L}_2[0,\infty)} \|z\|_2 < \gamma$ *if and only if*

$$\overline{\sigma}(D_{11}) < \gamma, \quad H_\infty \in dom(Ric), \quad X_\infty = Ric(H_\infty \geq 0.$$

Moreover, an optimal u is given by

$$u = F_{2\infty} x = -D_{12}^* D_{11} w_{\text{fiworst}} + \begin{bmatrix} D_{12}^* D_{11} & I \end{bmatrix} Fx,$$

and the worst w_{sfworst} is given by

$$w_{\text{sfworst}} = (\gamma^2 I - D_{11}^* D_{11})^{-1} \left\{ (D_{11}^* C_1 + B_1^* X_\infty) x + D_{11}^* D_{12} u \right\}.$$

Proof. The condition for part (a) can be shown in the same way as in Chapter 15 and is, in fact, the solution to the FI problem for the general case. We now prove the condition for part (b). It is not hard to see that $\overline{\sigma}(D_{11}) < \gamma$ is necessary since control u cannot feed back w directly, so the D_{11} term cannot be (partially) eliminated as in the FI problem. The conditions $H_\infty \in dom(Ric)$ and $X_\infty = Ric(H_\infty \geq 0$ can be easily seen as necessary since

$$\sup_{w\in B\mathcal{L}_2[0,\infty)} \min_{u\in\mathcal{L}_2[0,\infty)} \|z\|_2 \leq \min_{u\in\mathcal{L}_2[0,\infty)} \sup_{w\in B\mathcal{L}_2[0,\infty)} \|z\|_2.$$

It is easy to verify directly that the optimal control and the worst disturbance can be chosen in the given form. □

Define

$$\begin{aligned} R_0 &= I - D_{11}^* D_\perp D_\perp^* D_{11}/\gamma^2 \\ \tilde{R}_0 &= I + D_{12}^* D_{11}(\gamma^2 I - D_{11}^* D_{11})^{-1} D_{11}^* D_{12} \\ \hat{R}_0 &= I - D_{11}^* D_{11}/\gamma^2. \end{aligned}$$

Then it is easy to show that

$$\begin{aligned} \|z\|^2 - \gamma^2 \|w\|^2 + \frac{d}{dt}(x^* X x) &= \left\| u + D_{12}^* D_{11} w - \begin{bmatrix} D_{12}^* D_{11} & I \end{bmatrix} F x \right\|^2 \\ &\quad -\gamma^2 \left\| R_0^{1/2}(w - F_{1\infty} x) \right\|^2 \end{aligned}$$

if conditions in (a) are satisfied. On the other hand, we have

$$\|z\|^2 - \gamma^2 \|w\|^2 + \frac{d}{dt}(x^* X x) = \left\| \tilde{R}_0^{1/2}(u - F_{2\infty} x) \right\|^2 - \gamma^2 \left\| \hat{R}_0^{1/2}(w - w_{\text{sfworst}}) \right\|^2$$

if conditions in (b) are satisfied. Integrating both equations from $t = 0$ to ∞ with $x(0) = x(\infty) = 0$ gives

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 = \left\| u + D_{12}^* D_{11} w - \begin{bmatrix} D_{12}^* D_{11} & I \end{bmatrix} F x \right\|_2^2 - \gamma^2 \left\| R_0^{1/2}(w - F_{1\infty} x) \right\|_2^2$$

if conditions in (a) are satisfied, and

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 = \left\| \tilde{R}_0^{1/2}(u - F_{2\infty} x) \right\|_2^2 - \gamma^2 \left\| \hat{R}_0^{1/2}(w - w_{\text{sfworst}}) \right\|_2^2$$

if conditions in (b) are satisfied. These relations suggest that an optimal control law and a worst disturbance for problem (a) are

$$u = -D_{12}^* D_{11} w + \begin{bmatrix} D_{12}^* D_{11} & I \end{bmatrix} F x, \quad w = F_{1\infty} x$$

and that an optimal control law and a worst disturbance for problem (b) are

$$u = F_{2\infty} x \quad w = w_{\text{sfworst}}.$$

Moreover, if problem (b) has a solution for a given γ, then the corresponding differential game problem

$$\min_{u\in\mathcal{L}_2[0,\infty)} \sup_{w\in\mathcal{L}_2[0,\infty)} \left\{\|z\|_2^2 - \gamma^2\|w\|_2^2\right\}$$

has a saddle point, i.e,

$$\sup_{w\in\mathcal{L}_2[0,\infty)} \min_{u\in\mathcal{L}_2[0,\infty)} \left\{\|z\|_2^2 - \gamma^2\|w\|_2^2\right\} = \min_{u\in\mathcal{L}_2[0,\infty)} \sup_{w\in\mathcal{L}_2[0,\infty)} \left\{\|z\|_2^2 - \gamma^2\|w\|_2^2\right\}$$

since the solvability of problem (b) implies the solvability of problem (a). However, the converse may not be true. In fact, it is easy to construct an example so that the problem (a) has a solution for a given γ and problem (b) does not. On the other hand, the problems (a) and (b) are equivalent if $D_{11} = 0$.

17.9 Parameterization of State Feedback $\mathcal{H}_\infty$ Controllers

In this section, we shall consider the parameterization of all state feedback control laws. We shall first assume for simplicity that $D_{11} = 0$ and show later how to reduce the general $D_{11} \neq 0$ case to an equivalent problem with $D_{11} = 0$. We shall assume

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right].$$

Note that the state feedback $\mathcal{H}_\infty$ problem is *not* a special case of the output feedback problem since $D_{21} = 0$. Hence the parameterization cannot be obtained from the output feedback.

Theorem 17.7 *Suppose that the assumptions $(AS1) - (AS3)$ are satisfied and that B_1 has the following SVD:*

$$B_1 = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*, \quad UU^* = I_n, \quad V^*V = I_{m_1}, \quad 0 < \Sigma \in \mathbb{R}^{r\times r}.$$

There exists an admissible controller $K(s)$ for the SF problem such that $\|T_{zw}\|_\infty < \gamma$ if and only if $H_\infty \in dom(Ric)$ and $X_\infty = Ric(H_\infty) \geq 0$. Furthermore, if these conditions are satisfied, then all admissible controllers satisfying $\|T_{zw}\|_\infty < \gamma$ can be parameterized as

$$K = F_\infty + \left\{ I_{m_2} + Q \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I_{m_1-r} \end{bmatrix} U^{-1}B_2 \right\}^{-1} Q \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I_{m_1-r} \end{bmatrix} U^{-1}(sI - \hat{A})$$

where $F_\infty = -(D_{12}^*C_1 + B_2^*X_\infty)$, $\hat{A} = A + \gamma^{-2}B_1B_1^*X_\infty + B_2F_\infty$, and $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \in \mathcal{RH}_2$ with $\|Q_1\|_\infty < \gamma$. The dimensions of Q_1 and Q_2 are $m_2 \times r$ and $m_2 \times (m_1 - r)$, respectively.