7
H_∞ Synthesis

In this chapter we consider optimal synthesis with respect to the H_∞ norm introduced in Chapter 3. Again we are concerned with the feedback arrangement of Figure 6.1 where we have two state space systems G and K, each having their familiar role.

We will pursue the answer to the following question: does there exist a state space controller K such that

- The closed loop system is internally stable;
- The closed loop performance satisfies
$$||S(\hat{G}, \hat{K})||_\infty < 1 \ .$$

Thus we only plan to consider the problem of making the closed loop contractive in the sense of H_∞. It is clear, however, that determining whether there exists a stabilizing controller so that $||S(\hat{G}, \hat{K})||_\infty < \gamma$, for some constant γ, can be achieved by rescaling the γ dependent problem to arrive at the contractive version given above. Furthermore, by searching over γ, our approach will allow us to get as close to the minimal H_∞ norm as we desire, but in contrast to our work on H_2 optimal control, we will not seek a controller that exactly optimizes the H_∞ norm.

There are many approaches for solving the H_∞ control problem. Probably the most celebrated solution is in terms of Riccati equations of a similar style to the H_2 solution of Chapter 6. Here we will present a solution based entirely on linear matrix inequalities, which has the main advantage that it can be obtained with relatively straightforward matrix tools, and without any restrictions on the problem data. In fact Riccati equations and LMIs are intimately related, an issue we will explain when proving the Kalman-Yakubovich-Popov lemma concerning the *analysis* of the H_∞ norm of a system, which will be key to the subsequent synthesis solution.

Before getting into the details of the problem, we make a few comments about the motivation for this optimization.

As discussed in Chapter 3, the H_∞ norm is the L_2-induced norm of a causal, stable, linear-time invariant system. More precisely, given a causal linear time-invariant operator $G : L_2(-\infty, \infty) \to L_2(-\infty, \infty)$, the corresponding operator in the isomorphic space $\hat{L}_2(j\mathbb{R})$ is a multiplication operator $M_{\hat{G}}$ for a certain $\hat{G}(s) \in H_\infty$, and

$$||G||_{L_2 \to L_2} = ||M_{\hat{G}}||_{\hat{L}_2 \to \hat{L}_2} = ||\hat{G}||_\infty$$

What is the motivation for minimizing such an induced norm? If we refer back to the philosophy of "making error signals small" discussed in Chapter 6, we are minimizing the maximum "gain" of the system in the energy or L_2 sense. Equivalently, the excitation w is considered to be an arbitrary L_2 signal and we wish to minimize its *worst-case* effect on the energy of z. This may be an appropriate criterion if, as opposed to the situation of Chapter 6, we know little about the spectral characteristics of w. We will discuss, in more detail, alternatives and tradeoffs for noise modeling in Chapter 9.

There is however a more important reason than noise rejection that motivates an induced norm criterion; as seen in Section 3.1.2, a contractive operator Q has the property that the invertibility of $I - Q$ is ensured; this so-called *small-gain* property will be key to ensuring stability of certain feedback systems, in particular when some of the components are not precisely specified. This reason has made H_∞ control a central subject in control theory; further discussion of this application is given later in the course.

7.1 Two important matrix inequalities

The entire synthesis approach of the chapter revolves around the two technical results presented here. The first of these is a result purely about matrices; the second is an important systems theory result and is frequently called the Kalman-Yacubovich-Popov lemma, or KYP lemma for short.

We begin by stating the following which the reader can prove as an exercise.

Lemma 7.1. *Suppose $\bar{P}$ and $\bar{Q}$ are matrices satisfying* $\ker \bar{P} = 0$ *and* $\ker \bar{Q} = 0$. *Then for every matrix Y there exists a solution J to*

$$\bar{P}^* J \bar{Q} = Y.$$

The above lemma is used to prove the next one which is one of the two major technical results of this section.

Lemma 7.2. *Suppose*

(a) P, Q and H are matrices and that H is symmetric;

(b) The matrices W_P and W_Q are full rank matrices satisfying $ImW_P = \ker P$ and $ImW_Q = \ker Q$.

Then there exists a matrix J such that

$$H + P^*J^*Q + Q^*JP < 0 \;, \tag{7.1}$$

if and only if, the inequalities

$$W_P^*HW_P < 0 \quad \text{and} \quad W_Q^*HW_Q < 0$$

both hold.

Observe that when the kernels of P and Q are not both nonzero the result does not apply as stated. However it is readily seen from Lemma 7.1, that if both of the kernels are zero then there is always a solution J. If for example only $\ker P = 0$ then $W_Q^*HW_Q < 0$ is a necessary and sufficient condition for a solution to (7.1) to exist, as follows by simplified version of the following proof.

Proof. We will show the equivalence of the conditions directly by construction. To begin define V_1 to be a matrix such that

$$\mathrm{Im} V_1 = \ker P \cap \ker Q,$$

and V_2 and V_3 such that

$$\mathrm{Im}\begin{bmatrix} V_1 & V_2 \end{bmatrix} = \ker P \quad \text{and} \quad \mathrm{Im}\begin{bmatrix} V_1 & V_3 \end{bmatrix} = \ker Q.$$

Without loss of generality we assume that V_1, V_2 and V_3 have full column rank and define V_4 so that

$$V = \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \end{bmatrix}$$

is square and nonsingular. Therefore the LMI (7.1) above holds, if and only if

$$V^*HV + V^*P^*J^*QV + V^*Q^*JPV < 0 \quad \text{does.} \tag{7.2}$$

Now PV and QV are simply the matrices P and Q on the domain basis defined by V; therefore they have the form

$$PV = \begin{bmatrix} 0 & 0 & P_1 & P_2 \end{bmatrix} \quad \text{and} \quad QV = \begin{bmatrix} 0 & Q_1 & 0 & Q_2 \end{bmatrix};$$

we also define the block components

$$V^*HV =: \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{12}^* & H_{22} & H_{23} & H_{24} \\ H_{13}^* & H_{23}^* & H_{33} & H_{34} \\ H_{14}^* & H_{24}^* & H_{34}^* & H_{44} \end{bmatrix}.$$

Further define the variable Y by

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} P_1^* \\ P_2^* \end{bmatrix} J^* \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}.$$

From their definitions $\ker \begin{bmatrix} P_1 & P_2 \end{bmatrix} = 0$ and $\ker \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = 0$, and so by Lemma 7.1 we see that Y is freely assignable by choosing an appropriate matrix J.

Writing out inequality (7.2) using the above definitions we get

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{12}^* & H_{22} & H_{23} + Y_{11}^* & H_{24} + Y_{21}^* \\ H_{13}^* & H_{23}^* + Y_{11} & H_{33} & H_{34} + Y_{12} \\ H_{14}^* & H_{24}^* + Y_{21} & H_{34}^* + Y_{12}^* & H_{44} + Y_{22} + Y_{22}^* \end{bmatrix} < 0.$$

Apply the Schur complement formula to the upper 3×3 block, and we see the above holds, if and only if, the two following inequalities are met.

$$\bar{H} := \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12}^* & H_{22} & H_{23} + Y_{11}^* \\ H_{13}^* & H_{23}^* + Y_{11} & H_{33} \end{bmatrix} < 0$$

$$\text{and} \quad H_{44} + Y_{22} + Y_{22}^* - \begin{bmatrix} H_{14} \\ H_{24} + Y_{21}^* \\ H_{34} + Y_{12} \end{bmatrix}^* \bar{H}^{-1} \begin{bmatrix} H_{14} \\ H_{24} + Y_{21}^* \\ H_{34} + Y_{12} \end{bmatrix} < 0$$

As already noted above Y is freely assignable and so we see that provided the first inequality can be achieved by choosing Y_{11}, the second can always be met by appropriate choice of Y_{12}, Y_{21} and Y_{22}. That is the above two inequalities can be achieved, if and only if, $\bar{H} < 0$ holds for some Y_{11}. Now applying a Schur complement on $\bar{H}$ with respect to H_{11}, we obtain

$$\begin{bmatrix} H_{11} & 0 & 0 \\ 0 & H_{22} - H_{12}^* H_{11}^{-1} H_{12} & Y_{11}^* + X^* \\ 0 & Y_{11} + X & H_{33} - H_{13}^* H_{11}^{-1} H_{13} \end{bmatrix} < 0,$$

where $X = H_{23}^* - H_{13}^* H_{11}^{-1} H_{12}$. Now since Y_{11} is freely assignable we see readily that the last condition can be satisfied, if and only if, the diagonal entries of the left hand matrix are all negative definite. Using the Schur complement result twice these three conditions can be converted to the equivalent conditions

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} < 0 \quad \text{and} \quad \begin{bmatrix} H_{11} & H_{13} \\ H_{13}^* & H_{33} \end{bmatrix} < 0.$$

By the choice of our basis we see that these hold, if and only if, $W_P^* H W_P < 0$ and $W_Q^* H W_Q < 0$ are both met. ■

Having proved this matrix result we move on to our second result, the KYP lemma.

7.1.1 The KYP Lemma

There are many versions of this result, which establishes the equivalence between a frequency domain inequality and a state-space condition in terms of either a Riccati equation or an LMI. The version given below turns an H_∞ norm condition into an LMI. Being able to do this is very helpful for attaining our goal of controller synthesis, however it is equally important simply as a finite dimensional analysis test for transfer functions.

Lemma 7.3. *Suppose $\hat{M}(s) = C(Is - A)^{-1}B + D$. Then the following are equivalent conditions.*

(i) The matrix A is Hurwitz and

$$\|\hat{M}\|_\infty < 1 \; ;$$

(ii) There exists a matrix $X > 0$ such that

$$\begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^*X + XA & XB \\ B^*X & -I \end{bmatrix} < 0 \, . \tag{7.3}$$

The condition in (ii) is clearly an LMI and gives us a very convenient way to evaluate the H_∞ norm of a transfer function. In the proof below we see proving that condition (ii) implies that (i) holds is reasonably straightforward, and involves showing the direct connection between the above LMI and the state space equations that describe M. Proving the converse is considerably harder; fortunately we will be able to exploit the Riccati equation techniques which were introduced in Chapter 6. An alternative proof, which employs only matrix arguments, will be given later in the course.

Proof. We begin by showing (ii) implies (i). The top left block in (7.3) states that $A^*X + XA + C^*C < 0$. Since $X > 0$ we see that A must be Hurwitz.

It remains to show contractiveness which we do by employing a system-theoretic argument based on the state equations for M. Using the strict inequality (7.3) choose $1 > \epsilon > 0$ such that

$$\begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A^*X + XA & XB \\ B^*X & -(1-\epsilon)I \end{bmatrix} < 0 \tag{7.4}$$

holds. Let $w \in L_2[0, \infty)$ and realize that in order to show that M is contractive, it is sufficient to show that $\|z\|_2 \le (1-\epsilon)\|w\|_2$, where $z := Mw$. The state space equations relating w and z are

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \quad x(0) = 0, \\ z(t) &= Cx(t) + Dw(t). \end{aligned}$$

Now multiplying inequality (7.4) on the left by $\begin{bmatrix} x^*(t) & w^*(t) \end{bmatrix}$ and on the right by the adjoint we have

$$|z(t)|_2^2 + x^*(t)X(Ax(t) + Bw(t)) + (Ax(t) + Bw(t))^*Xx(t) - (1-\epsilon)|w(t)|_2^2 \le 0$$

By introducing the storage function $V : \mathbb{R}^n \to \mathbb{R}$, defined by $V(x(t)) = x^*(t)Xx(t)$, we arrive at the so-called dissipation inequality

$$\dot{V} + |z(t)|_2^2 \leq (1-\epsilon)|w(t)|_2^2.$$

Integrating on an interval $[0, T]$, recalling that $x(0) = 0$, gives

$$x(T)^*Xx(T) + \int_0^T |z(t)|_2^2 dt \leq (1-\epsilon)\int_0^T |w(t)|_2^2 dt.$$

Now let $T \to \infty$; since $w \in L_2$ and A is Hurwitz, then $x(T)$ converges to zero and therefore we find

$$||z||_2^2 \leq (1-\epsilon)||w||_2^2,$$

which completes this direction of the proof.

We now tackle the direction (i) implies (ii). To simplify the expressions we will write the derivation in the special case $D = 0$, but an analogous argument applies to the general case (see the exercises). Starting from

$$\hat{M}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right],$$

and recalling the definition of $\hat{M}^{\sim}(s)$ from Chapter 6, we derive the state-space representation

$$I - \hat{M}^{\sim}(s)\hat{M}(s) = \left[\begin{array}{cc|c} A & 0 & -B \\ -C^*C & -A^* & 0 \\ \hline 0 & B^* & I \end{array}\right].$$

It is easy to verify that

$$[I - \hat{M}^{\sim}(s)\hat{M}(s)]^{-1} = \left[\begin{array}{cc|c} A & BB^* & B \\ -C^*C & -A^* & 0 \\ \hline 0 & B^* & I \end{array}\right]. \tag{7.5}$$

Since $||\hat{M}||_\infty < 1$ by hypothesis, we conclude that $[I - \hat{M}^{\sim}(s)\hat{M}(s)]^{-1}$ has no poles on the imaginary axis. Furthermore we now show, using the PBH test, that the realization (7.5) has no unobservable eigenvalues that are purely imaginary. Suppose that

$$\begin{bmatrix} j\omega_0 I - A & -BB^* \\ C^*C & j\omega_0 I + A^* \\ 0 & B^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$

for some vectors x_1 and x_2. Then we have the following chain of implications:

$$\begin{aligned} B^*x_2 = 0 \ \ & \text{implies } (j\omega_0 I - A)x_1 = 0; \\ & \text{therefore } \ x_1 = 0 \quad \text{since } A \text{ is Hurwitz;} \\ & \text{this means } \ (j\omega_0 I + A^*)x_2 = 0; \\ & \text{which implies } x_2 = 0 \quad \text{again because } A \text{ is Hurwitz.} \end{aligned}$$

We conclude that (7.5) has no unobservable eigenvalues on the imaginary axis; an analogous argument shows the absence of uncontrollable eigenvalues. This means that the matrix

$$H = \begin{bmatrix} A & BB^* \\ -C^*C & -A^* \end{bmatrix}$$

has no purely imaginary eigenvalues. Referring to Theorem 5 in Chapter 6, notice that $BB^* \geq 0$ and (A, BB^*) is stabilizable since A is Hurwitz. Hence H is in the domain of the Riccati operator, and we can define $X_0 = \mathrm{Ric}(H)$ satisfying

$$A^*X_0 + X_0A + C^*C + X_0BB^*X_0 = 0 \tag{7.6}$$

and $A + BB^*X_0$ Hurwitz. Also note that (7.6) implies $A^*X_0 + X_0A \leq 0$, therefore from our work on Lyapunov equations we see that

$$X_0 \geq 0$$

since A is Hurwitz. To obtain the LMI characterization of (ii) we must slightly strengthen the previous relationships. For this purpose define $\bar{X}$ to be the solution of the Lyapunov equation

$$(A + BB^*X_0)^*\bar{X} + \bar{X}(A + BB^*X_0) = -I. \tag{7.7}$$

Since $(A + BB^*X_0)$ is Hurwitz we have $\bar{X} > 0$. Now let $X = X_0 + \epsilon\bar{X} > 0$, which is positive definite for all $\epsilon > 0$. Using (7.6) and (7.7) we have

$$A^*X + XA + C^*C + XBB^*X = -\epsilon I + \epsilon^2\bar{X}BB^*\bar{X}.$$

Choose $\epsilon > 0$ sufficiently small so that this equation is negative definite. Hence we have found $X > 0$ satisfying the strict *Riccati Inequality*

$$A^*X + XA + C^*C + XBB^*X < 0.$$

Now applying a Schur complement operation, this inequality is equivalent to

$$\begin{bmatrix} A^*X + XA + C^*C & XB \\ B^*X & -I \end{bmatrix} < 0\,,$$

which is (7.3) for the special case $D = 0$. ■

The preceding proof illustrates some of the deepest relationships of linear systems theory. We have seen that frequency domain inequalities are associated with dissipativity of storage functions in the time domain, and also the connection between LMIs (linked to dissipativity) and Riccati equations (which arise in quadratic optimization).

In fact this latter connection extends as well to problems of H_∞ synthesis, where both Riccati equations and LMIs can be used to solve the suboptimal control problem. In this course we will pursue the LMI solution. Surprisingly the two results of this section are all we require, together with basic matrix algebra, to solve our control problem.

7.2 Synthesis

We start with the state space realizations that describe the systems G and K:

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right], \quad \hat{K}(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right].$$

Notice that we have assumed $D_{22} = 0$. Removing this assumption leads to more complicated formulae, but the technique is identical. We make no other assumptions about the state space systems. The state dimensions of the nominal system and controller will be important: $A \in \mathbb{R}^{n \times n}$, $A_K \in \mathbb{R}^{n_K \times n_K}$.

Our first step is to combine these two state space realizations into one which describes the map from w to z. We obtain

$$S(\hat{G}, \hat{K}) = \left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right] = \left[\begin{array}{cc|c} A + B_2 D_K C_2 & B_2 C_K & B_1 + B_2 D_K D_{21} \\ B_K C_2 & A_K & B_K D_{21} \\ \hline C_1 + D_{12} D_K C_2 & D_{12} C_K & D_{11} + D_{12} D_K D_{21} \end{array}\right].$$

Now define the matrix

$$J = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix},$$

which collects the representation for K into one matrix. We can parametrize the closed-loop relation in terms of the controller realization as follows. First make the following definitions.

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} & \bar{B} &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\ \bar{C} &= \begin{bmatrix} C_1 & 0 \end{bmatrix} & \underline{C} &= \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix} \\ \underline{B} &= \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix} & \underline{D}_{12} &= \begin{bmatrix} 0 & D_{12} \end{bmatrix} \\ \underline{D}_{21} &= \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} \end{aligned} \tag{7.8}$$

which are entirely in terms of the state space matrices for G. Then we have

$$\begin{aligned} A_L &= \bar{A} + \underline{B} J \underline{C} & B_L &= \bar{B} + \underline{B} J \underline{D}_{21} \\ C_L &= \bar{C} + \underline{D}_{12} J \underline{C} & D_L &= D_{11} + \underline{D}_{12} J \underline{D}_{21} \end{aligned} \tag{7.9}$$

The crucial point here is that the parametrization of the closed loop state space matrices is *affine* in the controller matrix J.

Now we are looking for a controller K such that the closed loop is contractive and internally stable. The following form of the KYP lemma will help us.

Corollary 7.4. *Suppose $\hat{M}_L(s) = C_L(Is - A_L)^{-1}B_L + D_L$. Then the following are equivalent conditions.*

(a) The matrix A_L is Hurwitz and $\|\hat{M}_L\|_\infty < 1$;

(b) There exists a symmetric positive definite matrix X_L such that

$$\begin{bmatrix} A_L^* X_L + X_L A_L & X_L B_L & C_L^* \\ B_L^* X_L & -I & D_L^* \\ C_L & D_L & -I \end{bmatrix} < 0 \,.$$

This result is readily proved from Lemma 7.3 by applying the Schur complement formula. Notice that the matrix inequality in (b) is affine in X_L and J individually, but it is not jointly affine in both variables. The main task now is to obtain a characterization where we do have a convex problem.

Now define the matrices

$$P_{X_L} = \begin{bmatrix} \underline{B}^* X_L & 0 & \underline{D}_{12}^* \end{bmatrix}$$
$$Q = \begin{bmatrix} \underline{C} & \underline{D}_{21} & 0 \end{bmatrix}$$

and further

$$H_{X_L} = \begin{bmatrix} \bar{A}^* X_L + X_L \bar{A} & X_L \bar{B} & \bar{C}^* \\ \bar{B}^* X_L & -I & D_{11}^* \\ \bar{C} & D_{11} & -I \end{bmatrix} .$$

It follows that the inequality in (b) above is exactly

$$H_{X_L} + Q^* J^* P_{X_L} + P_{X_L}^* J Q < 0 \,.$$

Lemma 7.5. *Given the above definitions there exists a controller synthesis K if and only if there exists a symmetric matrix $X_L > 0$ such that*

$$W_{P_{X_L}}^* H_{X_L} W_{P_{X_L}} < 0 \quad \textit{and} \quad W_Q^* H_{X_L} W_Q < 0 \,,$$

where $W_{P_{X_L}}$ and W_Q are as defined in Lemma 7.2.

Proof. From the discussion above we see that a controller K exists if and only if there exists $X_L > 0$ satisfying

$$H_{X_L} + Q^* J^* P_{X_L} + P_{X_L}^* J Q < 0 \,.$$

Now invoke Lemma 7.2.

∎

This lemma says that a controller exists if and only if the two matrix inequalities can be satisfied. Each of the inequalities is given in terms of the state space matrices of G and the variable X_L. However we must realize that since X_L appears in both H_{X_L} and $W_{P_{X_L}}$, that these are not LMI conditions. Converting to an LMI formulation is our next goal, and will require a number of steps. Given a matrix $X_L > 0$ define the related matrix

$$T_{X_L} = \begin{bmatrix} \bar{A}X_L^{-1} + X_L^{-1}\bar{A}^* & \bar{B} & X_L^{-1}\bar{C}^* \\ \bar{B}^* & -I & D_{11}^* \\ \bar{C}X_L^{-1} & D_{11} & -I \end{bmatrix}, \tag{7.10}$$

and the matrix

$$P = \begin{bmatrix} \underline{B}^* & 0 & \underline{D}_{12}^* \end{bmatrix} \tag{7.11}$$

which only depends on the state space realization of G. The next lemma converts one of the two matrix inequalities of the lemma, involving H_{X_L}, to one in terms of T_{X_L}.

Lemma 7.6. *Suppose* $X_L > 0$. *Then*

$$W_{P_{X_L}}^* H_{X_L} W_{P_{X_L}} < 0, \quad \textit{if and only if,} \quad W_P^* T_{X_L} W_P < 0.$$

Proof. Start by observing that

$$P_{X_L} = PS \ ,$$

where

$$S = \begin{bmatrix} X_L & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} .$$

Therefore we have

$$\ker P_{X_L} = S^{-1} \ker P \ .$$

Then using the definitions of $W_{P_{X_L}}$ and W_P we can set

$$W_{P_{X_L}} = S^{-1} W_P \ .$$

Finally we have that $W_{P_{X_L}}^* H_{X_L} W_{P_{X_L}} < 0$ if and only if

$$W_P^*(S^{-1})^* H_{X_L} S^{-1} W_P < 0$$

and it is routine to verify $(S^{-1})^* H_{X_L} S^{-1} = T_{X_L}$.

■

Combining the last two lemmas we see that there exists a controller of state dimension n_K if and only if there exists a symmetric matrix $X_L > 0$ such that

$$W_P^* T_{X_L} W_P < 0 \quad \text{and} \quad W_Q^* H_{X_L} W_Q < 0 \ . \tag{7.12}$$

The first of these inequalities is an LMI in the matrix variable X_L^{-1}, where as the second is an LMI in terms of X_L. However the system of both inequalities is not an LMI. Our intent is to convert these seemingly non-convex conditions into an LMI condition.

Recall that X_L is a real and symmetric $(n+n_K)\times(n+n_K)$ matrix; here n and n_K are state dimensions of G and K. Let us now define the matrices X and Y which are $n\times n$ submatrices of X_L and X_L^{-1}, by

$$X_L =: \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} \quad \text{and} \quad X_L^{-1} =: \begin{bmatrix} Y & Y_2 \\ Y_2^* & Y_3 \end{bmatrix}. \tag{7.13}$$

We now show that the two inequality conditions listed in (7.12), only constrain the submatrices X and Y.

Lemma 7.7. *Suppose X_L is a positive definite $(n+n_K)\times(n+n_K)$ matrix and X and Y are $n\times n$ matrices defined as in (7.13). Then*

$$W_P^* T_{X_L} W_P < 0 \quad \textit{and} \quad W_Q^* H_{X_L} W_Q < 0\,,$$

if and only if, the following two matrix inequalities are satisfied

(a)

$$\begin{bmatrix} N_X & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A^*X+XA & XB_1 & C_1^* \\ B_1^*X & -I & D_{11}^* \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} N_X & 0 \\ 0 & I \end{bmatrix} < 0\,;$$

(b)

$$\begin{bmatrix} N_Y & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} AY+YA^* & YC_1^* & B_1 \\ C_1Y & -I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} \begin{bmatrix} N_Y & 0 \\ 0 & I \end{bmatrix} < 0\,,$$

where N_X and N_Y are full-rank matrices whose images satisfy

$$\begin{aligned} ImN_X &= \ker \begin{bmatrix} C_2 & D_{21} \end{bmatrix} \\ ImN_Y &= \ker \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}. \end{aligned}$$

Proof. The proof amounts to writing out the definitions and removing redundant constraints. Let us show that $W_P^* T_{X_L} W_P < 0$ is equivalent to the LMI in (b).

From the definitions of T_{X_L} in (7.10), and $\bar{A}$, $\bar{B}$ and $\bar{C}$ in (7.8) we get

$$T_{X_L} = \begin{bmatrix} AY+YA^* & AY_2 & B_1 & YC_1^* \\ Y_2^*A^* & 0 & 0 & Y_2^*C_1^* \\ B_1^* & 0 & -I & D_{11}^* \\ C_1Y & C_1Y_2 & D_{11} & -I \end{bmatrix}$$

Also recalling the definition of P in (7.11), and substituting for $\underline{B}$ and $\underline{D}_{12}$ from (7.8) yields

$$P = \begin{bmatrix} 0 & I & 0 & 0 \\ B_2^* & 0 & 0 & D_{12}^* \end{bmatrix}.$$

Thus the kernel of P is the image of

$$W_P = \begin{bmatrix} V_1 & 0 \\ 0 & 0 \\ 0 & I \\ V_2 & 0 \end{bmatrix}$$

where

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = N_Y$$

spans the kernel of $\begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}$ as defined above. Notice that the second block row of W_P is exactly zero, and therefore the second block-row and block-column of T_{X_L}, as explained above, do not enter into the constraint $W_P^* T_{X_L} W_P < 0$. Namely this inequality is

$$\begin{bmatrix} V_1 & 0 \\ 0 & I \\ V_2 & 0 \end{bmatrix}^* \begin{bmatrix} AY + YA^* & B_1 & YC_1^* \\ B_1^* & -I & D_{11}^* \\ C_1 Y & D_{11} & -I \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & I \\ V_2 & 0 \end{bmatrix} < 0\,.$$

By applying the permutation

$$\begin{bmatrix} V_1 & 0 \\ 0 & I \\ V_2 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} N_Y & 0 \\ 0 & I \end{bmatrix}$$

we arrive at (b).

Using a nearly identical argument, we can readily show that $W_Q^* H_{X_L} W_Q < 0$ is equivalent to LMI (a) in the theorem statement.

■

What we have shown is that a controller synthesis exists if and only if there exists an $(n+n_K) \times (n+n_K)$ matrix X_L that satisfies conditions (a) and (b) of the last lemma. These latter two conditions only involve X and Y, which are submatrices of X_L and X_L^{-1} respectively. Our next result tell us under what conditions, given arbitrary matrices X and Y, it is possible to find a positive definite matrix X_L that satisfies (7.13).

Lemma 7.8. *Suppose X and Y are symmetric, positive definite matrices in $\mathbb{R}^{n\times n}$; and n_K is a positive integer. Then there exist matrices $X_2, Y_2 \in \mathbb{R}^{n\times n_K}$ and symmetric matrices $X_3, Y_3 \in \mathbb{R}^{n_K \times n_K}$, satisfying*

$$\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} > 0 \quad \textit{and} \quad \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^* & Y_3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \quad \textit{and} \ \operatorname{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + n_K\,. \tag{7.14}$$

Proof. First we prove that the first two conditions imply the second two. From

$$\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} Y & Y_2 \\ Y_2^* & Y_3 \end{bmatrix} = I \tag{7.15}$$

it is routine to verify that

$$0 \leq \begin{bmatrix} I & 0 \\ Y & Y_2 \end{bmatrix} \begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & Y_2^* \end{bmatrix} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}.$$

Also the Schur complement relationship

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} = \begin{bmatrix} I & Y^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} X - Y^{-1} & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} I & 0 \\ Y^{-1} & I \end{bmatrix} \tag{7.16}$$

implies that

$$\operatorname{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} = n + \operatorname{rank}(X - Y^{-1}) = n + \operatorname{rank}(XY - I) \leq n + n_k,$$

where the last inequality follows from (7.15): $I - XY = X_2 Y_2^*$ and $X_2 \in \mathbb{R}^{n \times n_K}$.

To prove "if" we start with the assumption that (7.14) holds; therefore (7.16) gives

$$X - Y^{-1} \geq 0 \;\text{ and rank }\; (X - Y^{-1}) \leq n_K \ .$$

These conditions ensure that there exists a matrix $X_2 \in \mathbb{R}^{n \times n_K}$ so that

$$X - Y^{-1} = X_2 X_2^* \geq 0 \ .$$

From this and the Schur complement argument we see that

$$\begin{bmatrix} X & X_2 \\ X_2^* & I \end{bmatrix} > 0 \ .$$

Also

$$\begin{bmatrix} X & X_2 \\ X_2^* & I \end{bmatrix}^{-1} = \begin{bmatrix} Y & -YX_2 \\ -X_2^* Y & X_2^* Y X_2 + I \end{bmatrix}$$

and so we set $X_3 = I$.

■

The lemma states that a matrix X_L in $\mathbb{R}^{(n+n_K) \times (n+n_K)}$, satisfying (7.13), can be constructed from X and Y exactly when the LMI and rank conditions in (7.14) are satisfied. The rank condition is not in general an LMI, but notice that

$$\operatorname{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq 2n \ .$$

Therefore we see that if $n_K \geq n$ in the lemma, the rank condition becomes vacuous and we are left with only the LMI condition. We can now prove the synthesis theorem.

Theorem 7.9. *A synthesis exists for the H_∞ problem, if and only if there exist symmetric matrices $X > 0$ and $Y > 0$ such that*

(a)

$$\begin{bmatrix} N_X & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A^*X + XA & XB_1 & C_1^* \\ B_1^*X & -I & D_{11}^* \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} N_X & 0 \\ 0 & I \end{bmatrix} < 0\,;$$

(b)

$$\begin{bmatrix} N_Y & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} AY + YA^* & YC_1^* & B_1 \\ C_1Y & -I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} \begin{bmatrix} N_Y & 0 \\ 0 & I \end{bmatrix} < 0\,;$$

(c)

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0,$$

where N_X and N_Y are full-rank matrices whose images satisfy

$$\begin{aligned} ImN_X &= \ker \begin{bmatrix} C_2 & D_{21} \end{bmatrix} \\ ImN_Y &= \ker \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}\ . \end{aligned}$$

Proof. Suppose a controller exists, then by Lemma 7.7 a controller exists if and only if the inequalities

$$W_P^* T_{X_L} W_P < 0 \quad \text{and} \quad W_Q^* H_{X_L} W_Q < 0$$

hold for some symmetric, positive definite matrix X_L in $\mathbb{R}^{(n+n_K)\times(n+n_K)}$. By Lemma 7.7 these LMIs being satisfied imply that (a) and (b) are met. Also invoking Lemma 7.8 we see that (c) is satisfied.

Showing that (a–c) imply the existence of a synthesis is essentially the reverse process. We choose $n_K \geq n$, in this way the rank condition in Lemma 7.8 is automatically satisfied, and thus there exists an X_L in $\mathbb{R}^{(n+n_K)\times(n+n_K)}$ which satisfies (7.13). The proof is now completed by using X_L and (a–b) together with Lemma 7.7.

∎

This theorem gives us exact conditions under which a solution exists to our H_∞ synthesis problem. Notice that in the sufficiency direction we required that $n_k \geq n$, but clearly it suffices to choose $n_k = n$. In other words a synthesis exists if and only if one exists with state dimension $n_K = n$.

What if we want controllers of order n_K less than n? Then clearly from the above proof we have the following characterization.

Corollary 7.10. *A synthesis of order n_k exists for the H_∞ problem, if and only if there exist symmetric matrices $X > 0$ and $Y > 0$ satisfying (a), (b),* *and (c) in Theorem 7.9 plus the additional constraint*

$$\operatorname{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + n_K .$$

Unfortunately this constraint is not convex when $n_K < n$, so this says that in general the reduced order H_∞ problem is computationally much harder than the full order problem. Nevertheless, the above explicit condition can be exploited in certain situations.

7.3 Controller reconstruction

The results of the last section provide us with an explicit way to determine whether a synthesis exists which solves the H_∞ problem. Implicit in our development is a method to construct controllers when the conditions of Theorem 7.9 are met. We now outline this procedure, which simply retraces our steps so far.

Suppose X and Y have been found satisfying Theorem 7.9 then by Lemma 7.8 there exists a matrix $X_L \in \mathbb{R}^{n \times n_K}$ satisfying

$$X_L = \begin{bmatrix} X & ? \\ ? & ? \end{bmatrix} \text{ and } X_L^{-1} = \begin{bmatrix} Y & ? \\ ? & ? \end{bmatrix} .$$

From the proof of the lemma we can construct X_L by finding a matrix $X_2 \in \mathbb{R}^{n \times n_K}$ such that $X - Y^{-1} = X_2 X_2^*$. Then

$$X_L = \begin{bmatrix} X & X_2^* \\ X_2 & I \end{bmatrix}$$

has the properties desired above. As seen before, the order n_K need be no larger than n, and in general can be chosen to be the rank of $X - Y^{-1}$.

Next by Lemma 7.2 we know that there exists a solution to

$$H_{X_L} + Q^* J^* P_{X_L} + P_{X_L}^* J Q < 0 ,$$

and that any such solution J provides the state space realization for a feasible controller K. The solution of this LMI can be accomplished using standard techniques, and there is clearly an open set of solutions J.

7.4 Exercises

1. Prove Lemma 7.1.

2. Generalization of the KYP Lemma. Let A be a Hurwitz matrix, and let $\Psi = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$ be a symmetric matrix with $R > 0$. We define

$$\hat{\psi}(j\omega) = \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Psi \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}$$

Show that the following are equivalent:

(i) $\hat{\psi}(j\omega) \geq \epsilon > 0$ for all $\omega \in \mathbb{R}$.

(ii) The Hamiltonian matrix

$$H = \begin{bmatrix} A - BR^{-1}S^* & -BR^{-1}B^* \\ -(Q - BR^{-1}S^*) & -(A - BR^{-1}S^*)^* \end{bmatrix}$$

is in the domain of the Riccati operator.

(iii) The LMI

$$\begin{bmatrix} A^*X + XA & XB \\ B^*X & 0 \end{bmatrix} + \Psi > 0$$

admits a symmetric solution X.

(iv) There exists a quadratic storage function $V(x) = x^*Px$ such the dissipation inequality

$$\dot{V} \leq \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R - \epsilon I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

is satisfied over any solutions to the equation $\dot{x} = Ax + Bu$.

Hint: The method of proof from §7.1.1 can be replicated here.

3. Spectral Factorization. This exercise is a continuation of the previous one on the KYP Lemma. We take the same definitions for ψ, H, etc., and assume the above equivalent conditions are satisfied. Now set

$$\hat{M}(s) = \left[\begin{array}{c|c} A & B \\ \hline R^{-\frac{1}{2}}(S^* + B^*X) & R^{\frac{1}{2}} \end{array}\right],$$

where $X = Ric(H)$. Show that $\hat{M}(s) \in RH_\infty$, $\hat{M}(s)^{-1} \in RH_\infty$, and the factorization

$$\hat{\psi}(j\omega) = \hat{M}(j\omega)^* \hat{M}(j\omega)$$

holds for every $\omega \in \mathbb{R}$.

4. Mixed H_2/H_∞ control.

(a) We are given a stable system with the inputs partitioned in two channels w_1, w_2 and a common output z:

$$\hat{P} = [\hat{P}_1 \quad \hat{P}_2] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D & 0 \end{array}\right].$$

Suppose there exists $X > 0$ satisfying

$$\begin{bmatrix} A^*X + XA & XB_1 & C^* \\ B_1^*X & -I & D^* \\ C & D & -I \end{bmatrix} < 0, \tag{7.17}$$

$$\mathrm{Tr}(B_2^*XB_2) < \gamma^2. \tag{7.18}$$

Show that P satisfies the specifications $\|\hat{P}_1\|_{H_\infty} < 1$, $\|\hat{P}_2\|_{H_2} < \gamma$. Is the converse true?

(b) We now wish to use part (a) for state-feedback synthesis. In other words, given an open loop system

$$\begin{aligned} \dot{x} &= A_0x + B_1w_1 + B_2w_2 + B_0u, \\ z &= C_0x + Dw_1 + D_0u, \end{aligned}$$

we want to find a state feedback $u = Fx$ such that the closed loop satisfies (7.17)-(7.18). Substitute the closed loop matrices into (7.17); does this give an LMI problem for synthesis?

(c) Now modify (7.17) to an LMI in the variable X^{-1}, and show how to replace (7.18) by *two* convex conditions in X^{-1} and an appropriately chosen slack variable Z.

(d) Use part (c) to obtain a convex method for mixed H_∞/H_2 state feedback synthesis.

5. As a special case of reduced order H_∞ synthesis, prove Theorem 4.20 on model reduction.

6. Prove Theorem 5.8 involving stabilization. *Hint:* Reproduce the steps of the H_∞ synthesis proof, using a Lyapunov inequality in place of the KYP Lemma.

7. Connections to Riccati solutions for the H_∞ problem. Let

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

satisfy the normalization conditions

$$D_{12}^*\begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} \quad \text{and} \quad D_{21}\begin{bmatrix} B_1^* & D_{21}^* \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}.$$

Notice that these (and $D_{11} = 0$) are part of the conditions we imposed in our solution to the H_2-optimal control in the previous chapter.

(a) Show that the H_∞ synthesis is equivalent to the feasibility of the LMIs $X > 0$, $Y > 0$ and

$$\begin{bmatrix} A^*X + XA + C_1^*C_1 - C_2^*C_2 & XB_1 \\ B_1^*X & -I \end{bmatrix} < 0,$$
$$\begin{bmatrix} AY + YA^* + B_1B_1^* - B_2B_2^* & YC_1^* \\ C_1Y & -I \end{bmatrix} < 0,$$
$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0.$$

(b) Now denote $Q = Y^{-1}$, $P = X^{-1}$. Convert the above conditions to the following:

$$A^*P + PA + C_1^*C_1 + P(B_1B_1^* - B_2B_2^*)P < 0,$$
$$AQ + QA^* + B_1B_1^* + Q(C_1^*C_1 - C_2^*C_2)Q < 0,$$
$$\rho(PQ) \leq 1$$

These are two *Riccati inequalities* plus a spectral radius coupling condition. Formally analogous conditions involving the corresponding Riccati *equations* can be obtained when the plant satisfies some additional technical assumptions. For details consult the references.

Notes and references

The H_∞ control problem was formulated in [152], and was motivated by the necessity for a control framework that could systematically incorporate errors in the plant model. At the time this had been a goal of control research for a number of years, and H_2 control seemed poorly suited for this task [22]. The main observation of [152] was that these requirements could be met by working in a Banach algebra such as H_∞, but not H_2 which lacks this structure. We will revisit this question in subsequent chapters.

The formulation of the H_∞ problem precipitated an enormous research effort into its solution. The initial activity was based on the parametrization of stabilizing controllers discussed in Chapter 5, which reduced the problem to approximation in analytic function space. This problem was solved in the multivariable case by a combination of function theory and state space methods, notably Riccati equations. For an extensive account of this approach to the H_∞ problem, see the book [41]. Recent extensions to infinite dimensional systems can be found in [38].

Ultimately, these efforts led to a solution to the H_∞ problem in terms of two Riccati equations and based entirely on state-space methods [24]; see the books [50, 155] for an extensive presentation of this approach and historical references. This solution has close ties to the theory of differential games (see [6]). For extensions of the Riccati equation method to distributed parameter systems, see [132].

One of the drawbacks of the Riccati equation theory was that it required unnecessary rank conditions on the plant; the ensuing research in removing such restrictions [16, 126] led to the use of Riccati inequalities [117, 119] which pointed in the direction of LMIs. Complete LMI solutions to the problem with no unnecessary system requirements appeared in [42, 89]; these papers form the basis for this chapter, particularly the presentation in [42].

The LMI solution has, however, other advantages beyond this regularity question. In the first place, a family of controllers is parameterized, in contrast to the Riccati solution which over-emphasizes the so-called "central" solution. This increased flexibility can be exploited to impose other desirable requirements on the closed loop; for a recent survey of these multi-objective problems see [120].

Also, the LMI solution has led to powerful generalizations: in [18] a more general version is solved where spatial constraints can be specified; [31] solves the time varying and periodic problems; finally, the extension of this solution to multi-dimensional systems forms the basis of Linear Parameter-Varying control, a powerful method for gain-scheduling design [89].