Exercise session 7

ν-gap metric. LMI approach to H_∞ control.

Reading Assignment

[Zhou] Sections 17.2-3 and [Dullerud & Paganini] Chapter 7

Exercises

E7.1 Calculate $\delta_\nu(P_0, P_z)$ and $\delta_\nu(P_0, P_\eta)$ for plants

$$P_0 = \frac{10(s^2+1)}{s^2(s^2+2)}, \quad P_z = \frac{10(s^2+1.1)}{s^2(s^2+2)}, \quad P_\eta = \frac{10(s^2+1)}{s^2(s^2+1.8)}$$

and conclude about a sensitivity of δ_ν to the variation in the imaginary axis zeros and poles. Explain the result from the Riemann sphere interpretation of δ_ν.

E7.2 Connections to Riccati solutions for the H_∞ problem. Let

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

satisfy the normalization conditions

$$D_{12}^*\,[C_1 \quad D_{12}] = [0 \quad I] \quad \text{and} \quad D_{21}\,[B_1^* \quad D_{21}^*] = [0 \quad I].$$

(a) Show that the H_∞ synthesis is equivalent to the feasibility of the LMIs $X > 0$, $Y > 0$ and

$$\begin{bmatrix} A^*X + XA + C_1^*C_1 - C_2^*C_2 & XB_1 \\ B_1^*X & -I \end{bmatrix} < 0,$$
$$\begin{bmatrix} AY + YA^* + B_1B_1^* - B_2B_2^* & YC_1^* \\ C_1Y & -I \end{bmatrix} < 0,$$
$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0.$$

(b) Now denote $P = Y^{-1}$, $Q = X^{-1}$. Convert the above conditions to the following:

$$A^*P + PA + C_1^*C_1 + P(B_1B_1^* - B_2B_2^*)P < 0,$$
$$AQ + QA^* + B_1B_1^* + Q(C_1^*C_1 - C_2^*C_2)Q < 0,$$
$$\rho(PQ) \leq 1.$$

These are two *Riccati inequalities* plus a spectral radius coupling condition. Formally analogous conditions involving the corresponding Riccati *equations* can be obtained when the plant satisfies some additional technical assumptions. For details consult the references in [Dullerud & Paganini] Chapter 7.

E7.3 Consider the plant

$$P = \left[\begin{array}{cc|cc} -4 & 25 & 0.8 & -1 \\ -10 & 29 & 0.9 & -1 \\ \hline 10 & -25 & 0 & 1 \\ 13 & 25 & 1 & 0 \end{array}\right].$$

Apply [Dullerud & Paganini, Theorem 7.9] to verify if there exists a K such that the lower linear fractional transformation

$$\|T_{zw}\|_\infty = \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\|_\infty < \gamma,$$

where $\gamma = 1$. If so, construct a K using the method described in [Dullerud & Paganini, Section 7.3].

E7.4 (Very optional) Synthesize an H_∞ optimal state-feedback controller through minimizing γ over $P \succ 0$ and Q such that

$$\begin{bmatrix} PA^T + AP + Y^T B_2^T + B_2 Y & B_1 & P & Y^T \\ B_1^T & -\gamma^2 I & 0 & 0 \\ P & 0 & -I & 0 \\ Y & 0 & 0 & -I \end{bmatrix} \prec 0,$$

by the use of, e.g., CVX - see `http://cvxr.com/cvx/`. Perform synthesis for

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hand-In problems:

H7.1 Continuation of hand-in problem from Exercise 6. Let a nominal plant be given by $P = \frac{s-1}{s(s+2)}$ and we select $W = k$, a constant, as the shaping function. Thus $P_s = \frac{k(s-1)}{s(s+2)}$.

1. Calculate $\delta_\nu(P, P_s)$ for 4 different choices of $W = k$

$$k = [.1\ 1\ 5\ 10].$$

2. Conclusions?

H7.2 Repeat E7.3 for $\gamma = 1.5$.