Robust Control

Lecture 7: ν-gap metric + H_∞ synthesis through LMIs

Carolina Bergeling

Lecture 7

1. ν-gap metric (Sections 17.2-17.3 in Zhou)
2. H_∞ control through linear matrix inequalities (LMIs) (Chapter 7 in Dullerud/Paganini)

ν-Gap Metric

$$\delta_\nu(P_1, P_2) = \begin{cases} \|(I+P_2P_2^*)^{-\frac{1}{2}}(P_1-P_2)(I+P_1^*P_1)^{-\frac{1}{2}}\|_\infty & \text{if } \det(I+P_2^*P_1) \neq 0 \text{ on } j\mathbb{R} \text{ and} \\ & \text{wno}\det(I+P_2^*P_1) + \eta(P_1) = \overline{\eta}(P_2). \\ 1 & \text{otherwise} \end{cases}$$

where $\overline{\eta}$ (η) is the number of closed (open) RHP poles and wno is winding number.

Example

Consider

$$P_1(s) = \frac{1}{s}, \quad P_2(s) = \frac{1}{s+0.1}.$$

We had

$$\|P_1 - P_2\|_\infty = \left\| \frac{0.1}{s(s+0.1)} \right\|_\infty = +\infty.$$

However,

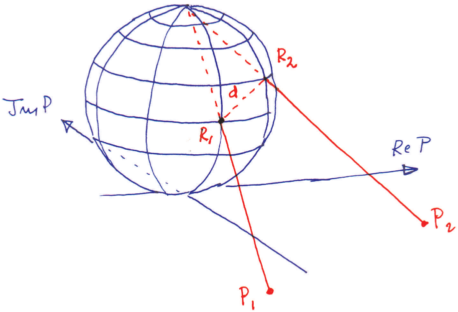
$$\delta_\nu(P_1, P_2) \approx 0.09951.$$

Geometric interpretation of ν-Gap Metric

In scalar case it takes on the particularly simple form

$$\delta_\nu(P_1, P_2) = \sup_{\omega \in R} \frac{|P_2(j\omega) - P_1(j\omega)|}{\sqrt{1 + |P_1(j\omega)|^2}\sqrt{1 + |P_2(j\omega)|^2}}$$

whenever the winding number condition is satisfied.



Some results

Theorem: Suppose (P_0, K) stable. Then, for any P

$$\arcsin b_{P,K} \geq \arcsin b_{P_0,K} - \arcsin \delta_\nu(P_0, P).$$

If $b_{P_0,K} > \delta_\nu(P_0, P)$ then (P, K) is stable.

Corollary: P_0 nominal plant and $\beta \leq \alpha < b_{opt}(P_0)$. For a given controller K,

$$\arcsin b_{P,K} \geq \arcsin \alpha - \arcsin \beta$$

for all P satisfying $\delta_\nu(P_0, P) \leq \beta$ if and only if $b_{P_0,K} > \alpha$.

New topic: H_∞ synthesis through LMIs

- The Kalman-Yakubovich-Popov Lemma
- H_∞ Optimal State Feedback
- The Matrix Elimination Lemma
- H_∞ Optimal Output feedback

The KYP Lemma

Suppose $\hat{M}(s) = C(sI - A)^{-1}B + D$. Then the following are equivalent conditions.

1. The matrix A is Hurwitz and $\|\hat{M}\|_\infty < \gamma$
2. There exists a matrix $X \succ 0$ such that

$$\begin{bmatrix} A^T X + XA & XB & C^T \\ B^T X & -\gamma^2 I & D^T \\ C & D & -I \end{bmatrix} \prec 0$$

The KYP Lemma is a classical result in systems theory connecting frequency domain to time domain. Many versions exist.

State feedback synthesis

Problem: Given $\dot{x} = Ax + B_1 w + B_2 u$, $x(0) = 0$, find control law $u = Kx$ such that the closed loop satisfies

$$\int_0^\infty |x|^2 + |u|^2 dt < \gamma^2 \int_0^\infty |w|^2.$$

Solution: Denote $z = \begin{bmatrix} x \\ u \end{bmatrix}$. Closed-loop system given by

$$\dot{x} = (A + B_2 K)x + B_1 w$$

$$z = \begin{bmatrix} I \\ K \end{bmatrix} x,$$

i.e, we demand $A + B_2 K$ to be Hurwitz and

$$\left\| \begin{bmatrix} I \\ K \end{bmatrix} (sI - (A + B_2 K))^{-1} B_1 \right\|_\infty < \gamma.$$

State feedback synthesis

Solution: KYP lemma says the following are equivalent

1. The matrix $A + B_2K$ is Hurwitz and $\left\| \begin{bmatrix} I \\ K \end{bmatrix} (sI - (A + B_2K))^{-1} B_1 \right\|_\infty < \gamma.$
2. There exists a matrix $X \succ 0$ such that

$$\begin{bmatrix} (A+B_2K)^T X + X(A+B_2K) & XB_1 & I & K^T \\ B_1^T X & -\gamma^2 I & 0 & 0 \\ I & 0 & -I & 0 \\ K & 0 & 0 & -I \end{bmatrix} \prec 0$$

We want to find $X \succ 0$ and K such that the above holds!

State feedback synthesis

Solution: Also equivalent to (left and right multiply inequality above with diag(X^{-1}, I, I, I) and set $P = X^{-1}$ and $Y = KX^{-1}$)

1. There exist matrices $P \succ 0$ and Y such that

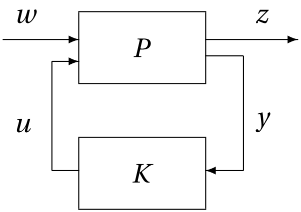
$$\begin{bmatrix} PA^T + AP + Y^T B_2^T + B_2 Y & B_1 & P & Y^T \\ B_1^T & -\gamma^2 I & 0 & 0 \\ P & 0 & -I & 0 \\ Y & 0 & 0 & -I \end{bmatrix} \prec 0.$$

This can be solved by convex optimization over P and Y. Finally, $K = YP^{-1}$.

Comparison to Riccati Approach

- More expensive to solve LMIs than AREs
- Fewer assumptions with LMI formulation
- Extensions possible at the expense of conservatism
  - Diagonal P and sparse Y gives sparse controller $K = YP^{-1}$
  - H_2 and H_∞ specifications can be merged

Output feedback synthesis



$$P: \begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w \end{cases}, \quad A \in \mathbb{R}^{n \times n}$$

($D_{22} \neq 0$ leads to more complicated formulae.)

$$K: \begin{cases} \dot{\xi} = A_K \xi + B_K y \\ u = C_K \xi + D_K y \end{cases}, \quad A_K \in \mathbb{R}^{n_K \times n_K}$$

Determine $J = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ to get $\int_0^\infty |z|^2 dt \leq \gamma^2 \int_0^\infty |w|^2 dt$

(γ set to 1 in following, to simplify notation)

Closed loop system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \underbrace{\begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x \\ \xi \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{bmatrix}}_{B_{cl}} w$$

$$z = \underbrace{\begin{bmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \end{bmatrix}}_{C_{cl}} + \underbrace{(D_{11} + D_{12} D_K D_{21})}_{D_{cl}} w$$

Notation

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \qquad \bar{C} = \begin{bmatrix} C_1 & 0 \end{bmatrix} \qquad \underline{\mathsf{C}} = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}$$

$$\underline{\mathsf{B}} = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix} \qquad \underline{\mathsf{D}}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} \qquad \underline{\mathsf{D}}_{12} = \begin{bmatrix} 0 & D_{21} \end{bmatrix}$$

Then

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & D_{11} \end{bmatrix} + \begin{bmatrix} \underline{\mathsf{B}} \\ \underline{\mathsf{D}}_{12} \end{bmatrix} J \begin{bmatrix} \underline{\mathsf{C}} & \underline{\mathsf{D}}_{21} \end{bmatrix}$$

and we define

$$\hat{M}_{cl}(s) = C_{cl}(sI - A_{cl})^{-1} B_{cl} + D_{cl}$$

LMI condition on the closed loop

It follows from the KYP Lemma that the following two conditions are equivalent

1. A_{cl} is Hurwitz and $\|\hat{M}_{cl}\|_\infty < 1$
2. There exists a matrix $X_{cl} \succ 0$ such that

$$\begin{bmatrix} A_{cl}^T X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^T \\ B_{cl}^T X_{cl} & -I & D_{cl}^T \\ C_{cl} & D_{cl} & -I \end{bmatrix} \prec 0$$

Notice: the inequality is affine in X_{cl} and J individually, but not jointly.

LMI condition rewritten

Notice that the inequality given previously can be written as

$$H_{X_{cl}} + Q^T J^T P_X + P_X^T J Q \prec 0$$

where

$$H_{X_{cl}} = \begin{bmatrix} \bar{A}^T X_{cl} + X_{cl}\bar{A} & X_{cl}\bar{B} & \bar{C}^T \\ \bar{B}^T X_{cl} & -I & D_{11}^T \\ \bar{C} & D_{11} & -I \end{bmatrix}, \qquad P_X = \begin{bmatrix} \underline{\mathsf{B}}^T C_{cl} & 0 & \underline{\mathsf{D}}_{12}^T \end{bmatrix} \qquad Q = \begin{bmatrix} \underline{\mathsf{C}} & \underline{\mathsf{D}}_{21} & 0 \end{bmatrix}.$$

The matrix elimination lemma

Given matrices P, Q, $H = H^T$ let N_p and N_Q be full rank matrices with $\mathrm{Im} N_P = \mathrm{Ker} P$ and $\mathrm{Im} N_Q = \mathrm{Ker} Q$. Then the following are equivalent

1. There exists J with $H + P^T J^T Q + Q^T J P \prec 0$
2. The inequalities $N_P^T H N_P \prec 0$ and $N_Q^T H N_Q \prec 0$ hold.

Output feedback continued

By the matrix elimination lemma, J exists that fulfils

$$H_{X_{cl}} + Q^T J^T P_X + P_X^T J Q \prec 0$$

if and only if

$$N_{P_X}^T H_{X_{cl}} N_{P_X} \prec 0 \quad \text{and} \quad N_Q^T H_{X_{cl}} N_Q \prec 0$$

The inequality $N_{P_X}^T H_{X_{cl}} N_{P_X} \prec 0$ can equialently be written

$$N_P^T T_{X_{cl}} N_P \prec 0$$

where

$$T_{X_{cl}} = \begin{bmatrix} \bar{A}^T X_{cl}^{-1} + X_{cl}^{-1}\bar{A} & \bar{B} & X_{cl}^{-1}\bar{C}^T \\ \bar{B}^T & -I & D_{11}^T \\ \bar{C}X_{cl}^{-1} & D_{11} & -I \end{bmatrix}, \qquad P = \begin{bmatrix} \underline{\mathsf{B}}^T & 0 & \underline{\mathsf{D}}_{12}^T \end{bmatrix}$$

(See Lemma 7.6 in Dullerud/Paganini)

Notice: $X_{cl} = \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}$ and $H_{X_{cl}}$ depend only on X, while $X_{cl}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}$ and $T_{X_{cl}}$ depend only on Y.

Lemma (Dullerud/Paganini Lemma 7.8)

Suppose $X \succ 0$ and $Y \succ 0$, $X, Y \in \mathbb{R}^{n \times n}$, and n_K is a positive integer, then the following are equivalent:

1. There exist $X_2, Y_2 \in \mathbb{R}^{n \times n_K}$ and symmetric matrices $X_3, Y_3 \in \mathbb{R}^{n_K \times n_K}$ such that

$$0 \prec \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1}$$

2. $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0$ and $\operatorname{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + n_K$

The rank condition goes away if $n_K = n$.

H_∞ output feedback synthesis (Dullerud/Paganini Th 7.9)

A controller that gives $\|\hat{M}_{cl}\|_\infty < 1$ exists if and only if there exist symmetric matrices $X \succ 0$ and $Y \succ 0$ such that

$$\begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB_1 & C_1^T \\ B_1^T X & -I & D_{11}^T \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + YA^T & YC_1^T & B_1 \\ C_1 Y & -I & D_{11} \\ B_1^T & D_{11}^T & -I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} \prec 0, \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0$$

where N_o, N_c are full rank matrices with Im$N_o = $ Ker$[C_2 \;\; D_{21}]$ and Im$N_c = $ Ker$[B_2^T \;\; D_{12}^T]$

Summary

- Robustness in terms of ν-gap metric
- H_∞ synthesis problems can be stated as convex optimization in terms of linear matrix inequalities
- Convexity requires $n_K = n$
- The H_∞ optimization involves coupling between estimation and control