LUNDS UNIVERSITET

Robust Control

Lecture 6: H_∞ loop shaping

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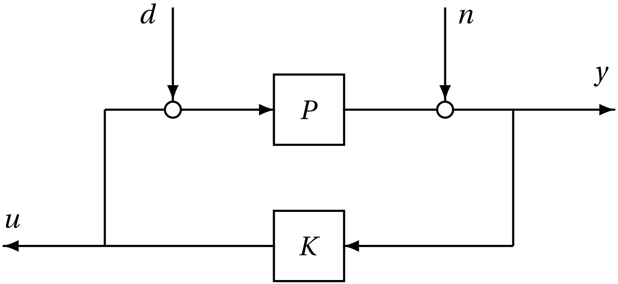
Lecture 6

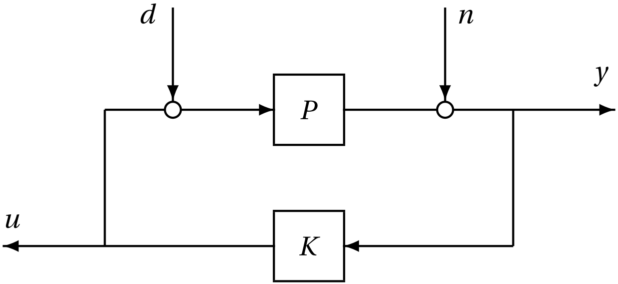
- An H_∞ Loop Shaping Procedure
- Properties of the robustness margin $b_{P,K}$
- Justification of H_∞ Loop Shaping.

What is Good Performance?



$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} n \\ d \end{bmatrix}$$

What is Good Performance?



What is captured by the norm

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty \quad ?$$

Remember: A controller should counteract disturbances, but be insensitive to measurement noise.

Loop-Shaping Design

Recall from Lecture 2 that a good performance controller design requires

- in the low frequency region:

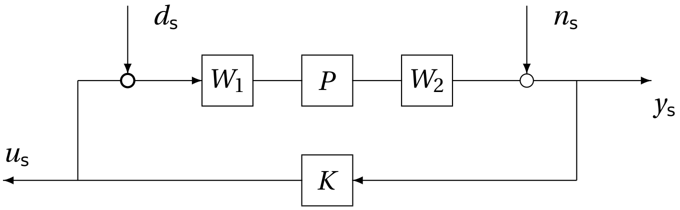
$$\underline{\sigma}(PK) >> 1, \quad \underline{\sigma}(KP) >> 1, \quad \underline{\sigma}(K) >> 1.$$

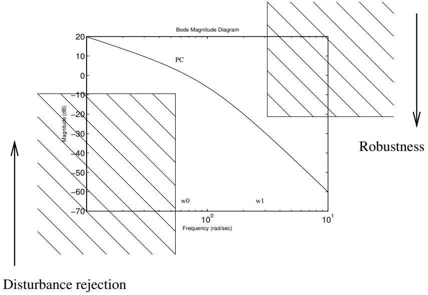
- in the high frequency region:

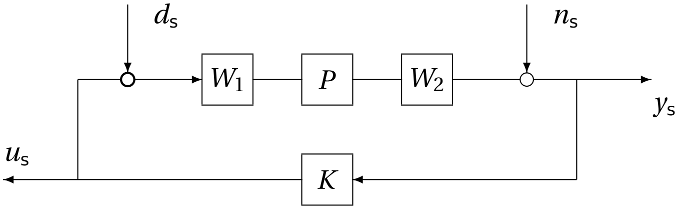
$$\overline{\sigma}(PK) << 1, \quad \overline{\sigma}(KP) << 1, \quad \overline{\sigma}(K) \leq M$$

where M is not too large.

Use weighting matrices!







1) Choose W_1 and W_2 and absorb them into the nominal plant P to get the shaped plant $P_s = W_2 P W_1$.
2) Design the controller K_∞ to minimize the H_∞ gain from (n_s, d_s) to (u_s, y_s). If the gain is large, the return to Step 1.
3) The final controller is $K = W_1 K_\infty W_2$.

(The H_∞ loop shaping design procedure was suggested by Glover and McFarlane, 1990.)

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- An H_∞ Loop Shaping Procedure.
- **Properties of the robustness margin $b_{P,K}$**
- Justification of H_∞ Loop Shaping.

A Notion of Loop Stability Margin

Introduce the quantity $b_{P,K}$

$$b_{P,K} = \begin{cases} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I+PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty^{-1} & \text{if } K \text{ stabilizes } P \\ 0 & \text{otherwise} \end{cases}$$

The larger $b_{P,K}$ is, the more robustly stable the closed loop system is.

Relation to Gain and Phase Margins

Theorem: Let P be a SISO plant and K be a stabilizing controller. Then

$$\begin{aligned} \text{gain margin} &\geq \frac{1+b_{P,K}}{1-b_{P,K}}, \\ \text{phase margin} &\geq 2\arcsin(b_{P,K}). \end{aligned}$$

Proof: For SISO system at every ω

$$b_{P,K} = \frac{1}{\|\ldots\|_\infty} \leq \frac{|1+P(j\omega)K(j\omega)|}{\left\| \begin{bmatrix} 1 \\ K \end{bmatrix} \begin{bmatrix} 1 & P \end{bmatrix} \right\|} = \frac{|1+P(j\omega)K(j\omega)|}{\sqrt{1+|P(j\omega)|^2}\sqrt{1+|K(j\omega)|^2}}$$

So at frequencies where $k := -PK \in R^+$ we have

$$
\begin{aligned}
b_{P,K} &\leq \frac{|1-k|}{\sqrt{(1+|P|^2)(1+k^2/|P|^2)}} \leq \\
&\leq \frac{|1-k|}{\sqrt{\min_P\{(1+|P|^2)(1+k^2/|P|^2)\}}} = \frac{|1-k|}{|1+k|}
\end{aligned}
$$

from which the gain margin result follows.

Similarly at frequencies where $PK = -e^{i\theta}$

$$
\begin{aligned}
b_{P,K} &\leq \frac{|1-e^{i\theta}|}{\sqrt{(1+|P|^2)(1+1/|P|^2)}} \leq \\
&\leq \frac{|1-e^{i\theta}|}{\sqrt{\min_P\{(1+|P|^2)(1+1/|P|^2)\}}} = \frac{2|\sin(\theta/2)|}{2}
\end{aligned}
$$

which implies the phase margin result.

Robust Stabilization of Coprime Factors

Let $P = \tilde{M}^{-1}\tilde{N}$, where $\tilde{N}(i\omega)\tilde{N}(i\omega)^* + \tilde{M}(i\omega)\tilde{M}(i\omega)^* \equiv 1$. This is called *normalized coprime factorization*.

The process $P_\Delta = (\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N)$ in feedback with the controller K is stable for all $\Delta = [\tilde{\Delta}_N \ \tilde{\Delta}_M]$ with $\|\Delta\|_\infty \leq \epsilon$ iff

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty < \frac{1}{\epsilon} \qquad (1)$$

Finding K that achieves (1) is a problem of H_∞ optimization. Lemma 16.4 (Zhou) shows equivalence between expression considered previously and the above.

H_∞ Optimization of Normalized Coprime Factors

Theorem: Let $D = 0$ and $L = -YC^*$ where $Y \geq 0$ is the stabilizing solution to $AY + YA^* - YC^*CY + BB^* = 0$. Then $P = \tilde{M}^{-1}\tilde{N}$ is a normalized left coprime factorization and

$$\begin{aligned}
\inf_{K-\text{stab}} \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1} \tilde{M}^{-1} \right\|_\infty &= \frac{1}{\sqrt{1 - \lambda_{\max}(YQ)}} \\
&= \left(1 - \| \tilde{N} \ \tilde{M} \|_H^2 \right)^{-1/2} = \gamma_{opt}
\end{aligned}$$

where Q is the solution to $Q(A - YC^*C) + (A - YC^*C)^*Q + C^*C = 0$. Moreover, a controller achieving $\gamma > \gamma_{opt}$ is

$$\begin{aligned}
K(s) &= \left(\begin{array}{c|c} A - BB^*X_\infty - YC^*C & -YC^* \\ \hline -B^*X_\infty & 0 \end{array} \right) \\
X_\infty &= \frac{\gamma^2}{\gamma^2 - 1} Q \left(I - \frac{\gamma^2}{\gamma^2 - 1} YQ \right)^{-1}
\end{aligned}$$

Right Coprime Factors

What if we have a normalized right coprime factorization $P = NM^{-1}$?

Theorem:

$$\left\| \begin{bmatrix} I \\ K \end{bmatrix} (I+PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\| \quad = \quad \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I+KP)^{-1} \begin{bmatrix} I & K \end{bmatrix} \right\|.$$

Corollary: Let $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be the normalized rcf and lcf, respectively. Then

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I+PK)^{-1} \tilde{M}^{-1} \right\|_\infty \quad = \quad \left\| M^{-1}(I+KP)^{-1} \begin{bmatrix} I & K \end{bmatrix} \right\|_\infty.$$

Conclusion: It does not matter what kind of factorization we have. One can work with either left or right.

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- Properties of the robustness margin $b_{P,K}$
- **Justification of H_∞ Loop Shaping.**

Loop-Shaping Design

Recall from Lecture 2 that a good performance controller design requires

- in the low frequency region:

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- in the high frequency region:

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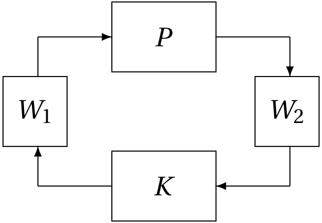
where M is not too large.

<u>Conclusion</u>: Performance depends strongly on open loop shape.



1) Choose W_1 and W_2 and absorb them into the nominal plant P to get the shaped plant $P_s = W_2 P W_1$.

2) Calculate $b_{opt}(P_s) = \sqrt{1 - \|\tilde{N}_s \ \tilde{M}_s\|_H^2}$. If it is small then return to Step 1 and adjust weights.

3) Select $\epsilon \le b_{opt}(P_s)$ and design the controller K_∞ such that

$$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I + P_s K_\infty)^{-1} \tilde{M}_s^{-1} \right\|_\infty < \epsilon^{-1}.$$

4) The final controller is $K = W_1 K_\infty W_2$.

Remarks:

- In contrast to the classical loop shaping design we do not treat explicitly closed loop stability, phase and gain margins. Thus the procedure is simple.
- Observe that

$$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I + P_s K_\infty)^{-1} \tilde{M}_s^{-1} \right\|_\infty = \left\| \begin{bmatrix} W_2 \\ W_1^{-1} K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} W_2^{-1} & PW_1 \end{bmatrix} \right\|_\infty$$

  so it has an interpretation of the standard H_∞ optimization problem with weights.
- BUT!!! The open loop under investigation on Step 1 is $K_\infty W_2 P W_1$ whereas the actual open loop is given by $W_1 K_\infty W_2 P$ and $P W_1 K_\infty W_2$. This is not really what we has shaped!

Thus the method needs validation.

Justification of H_∞ Loop Shaping

We show that the degradation in the loop shape caused by K_∞ is limited. Consider low-frequency region first.

$$\begin{aligned}\underline{\sigma}(PK) &= \underline{\sigma}(W_2^{-1}P_sK_\infty W_2) \geq \frac{\underline{\sigma}(P_s)\underline{\sigma}(K_\infty)}{\kappa(W_2)},\\ \underline{\sigma}(KP) &= \underline{\sigma}(W_1K_\infty P_sW_1^{-1}) \geq \frac{\underline{\sigma}(P_s)\underline{\sigma}(K_\infty)}{\kappa(W_1)}\end{aligned}$$

where κ denotes conditional number. Thus small $\underline{\sigma}(K_\infty)$ might cause problem even if P_s is large. Can this happen?

Theorem: Any K_∞ such that $b_{P_s,K_\infty} \geq 1/\gamma$ also satisfies

$$\underline{\sigma}(K_\infty) \geq \frac{\underline{\sigma}(P_s) - \sqrt{\gamma^2-1}}{\sqrt{\gamma^2-1}\,\underline{\sigma}(P_s)+1} \quad \text{if } \underline{\sigma}(P_s) > \sqrt{\gamma^2-1}.$$

Corollary: If $\underline{\sigma}(P_s) >> \sqrt{\gamma^2-1}$ then $\underline{\sigma}(K_\infty) \geq 1/\sqrt{\gamma^2-1}$

Consider now high frequency region.

$$\begin{aligned}\overline{\sigma}(PK) &= \overline{\sigma}(W_2^{-1}P_sK_\infty W_2) \le \overline{\sigma}(P_s)\overline{\sigma}(K_\infty)\kappa(W_2),\\ \overline{\sigma}(KP) &= \overline{\sigma}(W_1K_\infty P_sW_1^{-1}) \le \overline{\sigma}(P_s)\overline{\sigma}(K_\infty)\kappa(W_1).\end{aligned}$$

Can $\overline{\sigma}(K_\infty)$ be large if $\overline{\sigma}(P_s)$ is small?

Theorem: Any K_∞ such that $b_{P_s,K_\infty} \ge 1/\gamma$ also satisfies

$$\overline{\sigma}(K_\infty) \le \frac{\sqrt{\gamma^2-1}+\overline{\sigma}(P_s)}{1-\sqrt{\gamma^2-1}\,\overline{\sigma}(P_s)} \quad \text{if } \overline{\sigma}(P_s) < \frac{1}{\sqrt{\gamma^2-1}}.$$

Corollary: If $\overline{\sigma}(P_s) << 1/\sqrt{\gamma^2-1}$ then $\overline{\sigma}(K_\infty) \le \sqrt{\gamma^2-1}$

Denote

$$\overline{\sigma}_i = \overline{\sigma}(W_i), \quad \underline{\sigma}_i = \underline{\sigma}(W_i), \quad \kappa_i = \kappa(W_i).$$

Theorem: Let P be the nominal plant and let $K = W_1 K_\infty W_2$ be the controller designed by loop shaping. If $b_{P_s,K_\infty} \geq 1/\gamma$ then

$$\begin{aligned}
\overline{\sigma}(K(I+PK)^{-1}) &\leq \gamma\overline{\sigma}(\tilde{M}_s)\overline{\sigma}_1\overline{\sigma}_2, \\
\overline{\sigma}((I+PK)^{-1}) &\leq \min\{\gamma\overline{\sigma}(\tilde{M}_s)\kappa_2, 1+\gamma\overline{\sigma}(\tilde{N}_s)\kappa_2\}, \\
\overline{\sigma}(K(I+PK)^{-1}P) &\leq \min\{\gamma\overline{\sigma}(\tilde{N}_s)\kappa_1, 1+\gamma\overline{\sigma}(\tilde{M}_s)\kappa_1\}, \\
\overline{\sigma}((I+PK)^{-1}P) &\leq \frac{\gamma\overline{\sigma}(\tilde{N}_s)}{\underline{\sigma}_1\underline{\sigma}_2}, \\
\overline{\sigma}((I+KP)^{-1}) &\leq \min\{1+\gamma\overline{\sigma}(\tilde{N}_s)\kappa_1, \gamma\overline{\sigma}(\tilde{M}_s)\kappa_1\}, \\
\overline{\sigma}(P(I+KP)^{-1}K) &\leq \min\{1+\gamma\overline{\sigma}(\tilde{M}_s)\kappa_2, \gamma\overline{\sigma}(\tilde{N}_s)\kappa_2\}
\end{aligned}$$

where

$$\overline{\sigma}(\tilde{N}_s) = \left(\frac{\overline{\sigma}^2(P_s)}{1+\overline{\sigma}^2(P_s)}\right)^{1/2} \qquad \overline{\sigma}(\tilde{M}_s) = \left(\frac{1}{1+\overline{\sigma}^2(P_s)}\right)^{1/2}$$

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What have we learned today?

- H_∞ optimization of normalized coprime factors.
- Left or right coprime factors - does not matter.
- Stability margin $b_{P,K}$. The larger the better. Relation to gain and phase margins.
- H_∞ loop shaping via pre- and postcompensations and optimization of $b_{P,K}$.