Robust Control

Lecture 5: Algebraic Riccati Equations, H_2 and H_∞ control

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Summary so far

- Ch. 4: got to know the L_2, L_∞, H_2 and H_∞ spaces
- Ch. 5: amiliarized with the notion of *internal stability* (analysis through coprime factorization)
- Ch. 6: looked at performance specifications (singular values, weighted H_2 and H_∞ norms)
- Ch. 8: considered unstructured uncertainty models and robustness
- Ch. 9 : pulled out the Δ's and put system on general form through LFT
- Ch. 10: considered robust stability and performance under the μ framework (structured uncertainties)

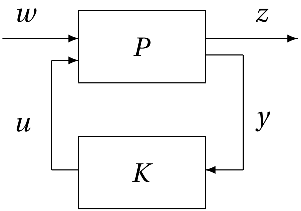
What is left in the course?

- Lecture 5 (Today's lecture!): Ch. 12, (13), 14: AREs, H_2 and H_∞ optimal control
- Lecture 6: Convex approach to H_∞ control (separate literature)
- Lecture 7: Ch. 16 and parts of 17: H_∞ Loop Shaping and the Bosse Show

Lecture 5

- The H_∞ Optimization Problem
- Linear Quadratic Games
- Algebraic Riccati Equations
- State Space Solution to H_∞ Optimization

The H_∞ Optimization Problem



$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$$T_{zw} = \mathscr{F}_l(P, K)$$

Optimal control:

$$\min_{K-\text{stab}} \|T_{zw}\|_\infty$$

Suboptimal control: Given γ find an internally stabilizing controller K such that

$$\|T_{zw}\|_\infty < \gamma.$$

The optimal control problem is solved by iterating γ in the suboptimal problem.

H_∞ Optimization in Frequency Domain

The *Youla parameterization* of all internally stabilizing controllers (see Chapter 11, if you are interested) gives an affine dependence of T_{zw} on the Youla parameter $Q \in RH_\infty$

$$T_{zw} = T_1 - T_2 Q T_3, \quad T_k \in RH_\infty$$

Thus the H_∞ optimization problem becomes

$$\min_{Q \in RH_\infty} \| T_1 - T_2 Q T_3 \|_\infty$$

The optimization in Q is convex, but infinite-dimensional.

State Space Solution: Recall LQ Control

If P satisfies the Riccati equation $A^T P + PA + Q - PBB^T P = 0$, then every solution to $\dot{x} = Ax + Bu$ with $\lim_{t\to\infty} x(t) = 0$ satisfies

$$\begin{aligned}
&\int_0^\infty [x^T Q x + u^T u] dt \\
&= \int_0^\infty |u + B^T P x|^2 dt - 2\int_0^\infty (Ax + Bu)^T P x dt \\
&= \int_0^\infty |u + B^T P x|^2 dt - 2\int_0^\infty \dot{x}^T P x dt \\
&= \int_0^\infty |u + B^T P x|^2 dt - \int_0^\infty \frac{d}{dt}[x^T P x] dt \\
&= \int_0^\infty |u + B^T P x|^2 dt + x(0)^T P x(0)
\end{aligned}$$

with the minimizing control law $u = -B^T P x$.

A Linear Quadratic Game

If X satisfies the Algebraic Riccati Equation

$$A^T X + XA + Q - X(B_u B_u^T - B_w B_w^T/\gamma^2)X \quad = \quad 0$$

then $\dot{x} = Ax + B_u u + B_w w$ with $x(0) = 0$ gives

$$\begin{aligned} &\int_0^\infty [x^T Q x + u^T u - \gamma^2 w^T w] dt \\ &= \int_0^\infty |u + B_u^T X x|^2 dt - \gamma^2 \int_0^\infty |w - B_w^T X x|^2 dt \end{aligned}$$

This can be viewed as a dynamic game between the player u, who tries to minimize and w who tries to maximize.

The minimizing control law $u = -B_u^T X x$ gives

$$\int_0^\infty [x^T Q x + u^T u] dt \leq \gamma^2 \int_0^\infty w^T w dt$$

so the gain from w to $z = (Q^{1/2} x, u)$ is at most γ.

Algebraic Riccati Equations

$$A^* X + XA + XRX + Q = 0$$

where $R = R^*$, $Q = Q^*$.

- The ARE is as important for control design as the Lyapunov equation is for system analysis.
- There are many solutions $X = X^*$ to ARE, the stabilizing one (which makes $A + RX$ stable) is unique!
- The ARE is a state space tool, which corresponds to factorization in frequency domain (recall spectral factorization in LQ Control).

How do we solve it?

Hamiltonian Matrix

Consider the $2n \times 2n$ matrix

$$H = \begin{pmatrix} A & R \\ -Q & -A^* \end{pmatrix}.$$

Lemma: Eigenvalues of H are symmetric with respect to the imaginary axis.

Proof: Introduce $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Then $J^{-1}HJ = -H^*$, so λ is an eigenvalue of H if and only if $-\bar{\lambda}$ is.

In particular, if there are no purely imaginary eigenvalues then there are precisely n stable and n unstable eigenvalues of H.

Stable Invariant Subspace

Under assumption of no purely imaginary eigenvalues, let

$$T = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in R^{2n \times n}$$

be a basis of the stable n-dimensional invariant subspace. Equivalently $HT = T\Lambda$ for some stable matrix $\Lambda \in R^{n \times n}$.

Lemma: If $\det(X_1) \neq 0$ then $X = X_2 X_1^{-1}$ is a stabilizing solution to the ARE $A^* X + XA + XRX + Q = 0$

Proof: We are to prove

1) $X = X^*$.

2) X satisfies the ARE.

3) $A + RX$ is stable.

1) $HT = T\Lambda \Rightarrow T^*JHT = T^*JT\Lambda$. The matrix JH is symmetric then

$$T^*JT\Lambda = \Lambda^*T^*J^*T \quad \Leftrightarrow \quad T^*JT\Lambda + \Lambda^*T^*JT = 0.$$

So T^*JT satisfies the Lyapunov equation and Λ is stable. Hence $T^*JT = 0$, that is

$$X_2^*X_1 - X_1^*X_2 = 0 \quad \Leftrightarrow \quad X^* - X = 0.$$

2) & 3) Simple calculation gives

$$\begin{aligned} AX_1 + RX_2 &= X_1\Lambda, \\ -QX_1 - A^*X_2 &= X_2\Lambda. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} A + RX &= X_1\Lambda X_1^{-1} \\ -Q - A^*X &= X_2\Lambda X_1^{-1}. \end{aligned}$$

Thus $A + RX$ is stable and

$$XA + XRX = X_2\Lambda X_1 = -Q - A^*X$$

which implies the ARE.

How to solve the ARE

Under conditions

(H1) There are no pure imaginary eigenvalues of H.

(H2) $\det(X_1) \neq 0$ for some basis of stable invariant subspace.

we can find a stabilizing solution to ARE as follows:

1. Find a basis T for the stable invariant subspace, for example by Schur decomposition. If **(H1)** holds, then it has the dimension n.
2. Partition T as
$$T = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$
(H2) holds for some basis iff it holds for all basis.
3. Build $X = X_2 X_1^{-1}$.

Notation

$H \in \text{dom}(\text{Ric})$ if (H1) and (H2) hold for H.

$X = \text{Ric}(H)$ is the stabilizing solution to ARE.

ARE for H_∞ Optimal State Feedback

Theorem: Consider $\dot{x} = Ax + B_u u + B_w w$, $x(0) = 0$, where (A, B_u) and (A, B_w) be stabilizable. Introduce the Hamiltonian

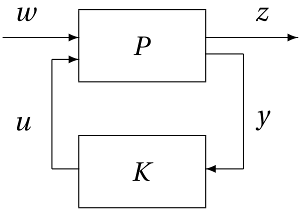
$$H_0 = \begin{pmatrix} A & B_w B_w^T/\gamma^2 - B_u B_u^T \\ -Q & -A^T \end{pmatrix}.$$

Then, the following conditions are equivalent:

1. There exists a stabilizing control law with $\int_0^\infty (x^T Q x + |u|^2) dt \leq \gamma^2 \int_0^\infty |w|^2 dt$
2. H_0 has no purely imaginary eigenvalues.
3. $H_0 \in \mathrm{dom(Ric)}$.

Proof: The implication (3) ⇒ (1) was proved on slide “A Linear Quadratic Game”. For (2) ⇒ (3), see [Zhou, p. 237].

Output Feedback Assumptions



$$P = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

(A1) (A, B_w, C_z) is stabilizable and detectable,

(A2) (A, B_u, C_y) is stabilizable and detectable,

(A3) $D_{zu}^* \begin{pmatrix} C_z & D_{zu} \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}$,

(A4) $\begin{pmatrix} B_w \\ D_{yw} \end{pmatrix} D_{yw}^* = \begin{pmatrix} 0 \\ I \end{pmatrix}$.

State Space H_∞ optimization

The solution involves two AREs with Hamiltonian matrices

$$H_\infty = \begin{pmatrix} A & \gamma^{-2}B_wB_w^* - B_uB_u^* \\ -C_z^*C_z & -A^* \end{pmatrix}$$

$$J_\infty = \begin{pmatrix} A^* & \gamma^{-2}C_z^*C_z - C_y^*C_y \\ -B_wB_w^* & -A \end{pmatrix}$$

Theorem: There exists a stabilizing controller K such that $\|T_{zw}\|_\infty < \gamma$ if and only if the following three conditions hold:

1. $H_\infty \in \mathrm{dom}(\mathrm{Ric})$ and $X_\infty = \mathrm{Ric}(H_\infty) \geq 0$,
2. $J_\infty \in \mathrm{dom}(\mathrm{Ric})$ and $Y_\infty = \mathrm{Ric}(J_\infty) \geq 0$,
3. $\rho(X_\infty Y_\infty) < \gamma^2$.

Moreover, one such controller is

$$K_{sub}(s) = \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$\hat{A}_\infty = A + \gamma^{-2} B_w B_w^* X_\infty + B_u F_\infty + Z_\infty L_\infty C_y,$$

$$F_\infty = -B_u^* X_\infty, \; L_\infty = -Y_\infty C_y^*,$$

$$Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

Furthermore, the set of all stabilizing controllers such that $\|T_{wz}\|_\infty < \gamma$ can be explicitly obtained as lower LFT (see [Zhou,p. 271]).

[Doyle J., Glover K., Khargonekar P., Francis B., *State Space Solution to Standard H^2 and H^∞ Control Problems*, IEEE Trans. on AC **34** (1989) 831–847.]

Idea of Proof

The dynamic game viewpoint gives a solution in the case of full information, where both state and disturbance are measured. This gives the first ARE.

This can be combined with a "worst case observer", finding the smallest disturbance compatible with available measurements. This gives the second ARE.

Combining the full information solution with the worst case observer, solves the dynamic game problem with limited measurement information, provided that the spectral radius condition holds.

H_2 optimal control

- LQR, state available for feedback. Guaranteed stability margins (see Ch 13.4).
- Standard H_2 problem/ LQG (output feedback): no guaranteed stability margins, remember example in first lecture

What have we learned today?

- H_∞ optimization is fundamental problem for robust synthesis.
- A dynamic game between controller and disturbance
- The state space approach gives easily implementable conditions and formulas.
- Algebraic Riccati Equation is the main computational tool.