Chapter 3

Basic Concepts

This chapter and the next are the most fundamental. We concentrate on the single-loop feedback system. Stability of this system is defined and characterized. Then the system is analyzed for its ability to track certain signals (i.e., steps and ramps) asymptotically as time increases. Finally, tracking is addressed as a performance specification. Uncertainty is postponed until the next chapter.

Now a word about notation. In the preceding chapter we used signals in the time and frequency domains; the notation was $u(t)$ for a function of time and $\hat{u}(s)$ for its Laplace transform. When the context is solely the frequency domain, it is convenient to drop the hat and write $u(s)$; similarly for an impulse response $G(t)$ and the corresponding transfer function $\hat{G}(s)$.

3.1 Basic Feedback Loop

The most elementary feedback control system has three components: a plant (the object to be controlled, no matter what it is, is always called the *plant*), a sensor to measure the output of the plant, and a controller to generate the plant's input. Usually, actuators are lumped in with the plant. We begin with the block diagram in Figure 3.1. Notice that each of the three components

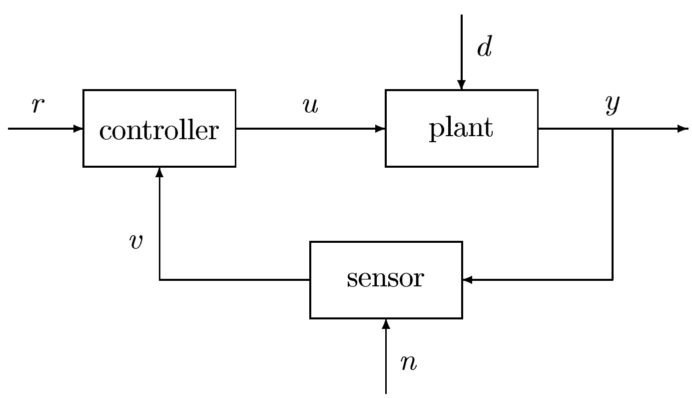


Figure 3.1: Elementary control system.

has two inputs, one internal to the system and one coming from outside, and one output. These signals have the following interpretations:

r	reference or command input
v	sensor output
u	actuating signal, plant input
d	external disturbance
y	plant output and measured signal
n	sensor noise

The three signals coming from outside—r, d, and n—are called *exogenous inputs.*

In what follows we shall consider a variety of performance objectives, but they can be summarized by saying that y should approximate some prespecified function of r, and it should do so in the presence of the disturbance d, sensor noise n, with uncertainty in the plant. We may also want to limit the size of u. Frequently, it makes more sense to describe the performance objective in terms of the measurement v rather than y, since often the only knowledge of y is obtained from v.

The analysis to follow is done in the frequency domain. To simplify notation, hats are omitted from Laplace transforms.

Each of the three components in Figure 3.1 is assumed to be linear, so its output is a linear function of its input, in this case a two-dimensional vector. For example, the plant equation has the form

$$y = P \begin{pmatrix} d \\ u \end{pmatrix}.$$

Partitioning the 1×2 transfer matrix P as

$$P = \begin{bmatrix} P_1 & P_2 \end{bmatrix},$$

we get

$$y = P_1 d + P_2 u.$$

We shall take an even more specialized viewpoint and suppose that the outputs of the three components are linear functions of the sums (or difference) of their inputs; that is, the plant, sensor, and controller equations are taken to be of the form

$$\begin{aligned} y &= P(d+u), \\ v &= F(y+n), \\ u &= C(r-v). \end{aligned}$$

The minus sign in the last equation is a matter of tradition. The block diagram for these equations is in Figure 3.2. Our convention is that plus signs at summing junctions are omitted.

This section ends with the notion of *well-posedness.* This means that in Figure 3.2 all closed-loop transfer functions exist, that is, all transfer functions from the three exogenous inputs to all internal signals, namely, u, y, v, and the outputs of the summing junctions. Label the outputs of the summing junctions as in Figure 3.3. For well-posedness it suffices to look at the nine transfer functions from r, d, n to x_1, x_2, x_3. (The other transfer functions are obtainable from these.) Write the equations at the summing junctions:

$$\begin{aligned} x_1 &= r - Fx_3, \\ x_2 &= d + Cx_1, \\ x_3 &= n + Px_2. \end{aligned}$$

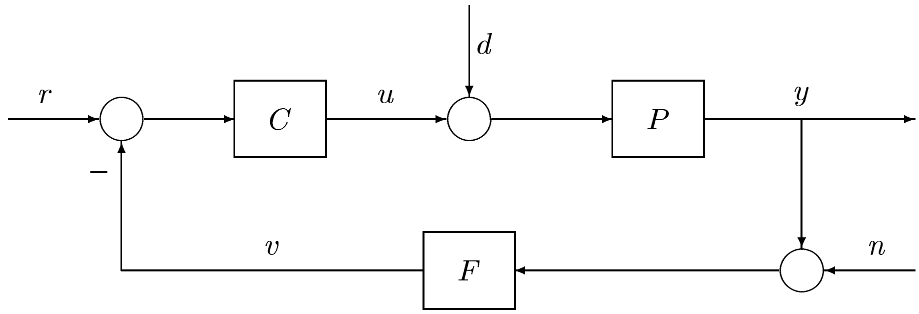


Figure 3.2: Basic feedback loop.

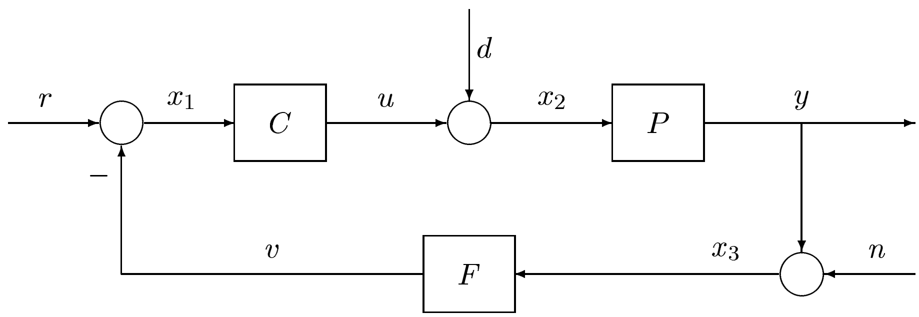


Figure 3.3: Basic feedback loop.

In matrix form these are

$$\begin{bmatrix} 1 & 0 & F \\ -C & 1 & 0 \\ 0 & -P & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} r \\ d \\ n \end{pmatrix}.$$

Thus, the system is well-posed iff the above 3×3 matrix is nonsingular, that is, the determinant $1 + PCF$ is not identically equal to zero. [For instance, the system with $P(s) = 1$, $C(s) = 1$, $F(s) = -1$ is not well-posed.] Then the nine transfer functions are obtained from the equation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & F \\ -C & 1 & 0 \\ 0 & -P & 1 \end{bmatrix}^{-1} \begin{pmatrix} r \\ d \\ n \end{pmatrix},$$

that is,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{1 + PCF} \begin{bmatrix} 1 & -PF & -F \\ C & 1 & -CF \\ PC & P & 1 \end{bmatrix} \begin{pmatrix} r \\ d \\ n \end{pmatrix}. \tag{3.1}$$

A stronger notion of well-posedness that makes sense when P, C, and F are proper is that the nine transfer functions above are proper. A necessary and sufficient condition for this is that $1 + PCF$ not be strictly proper [i.e., $PCF(\infty) \neq -1$].

One might argue that the transfer functions of all physical systems are strictly proper: If a sinusoid of ever-increasing frequency is applied to a (linear, time-invariant) system, the amplitude of the output will go to zero. This is somewhat misleading because a real system will cease to behave linearly as the frequency of the input increases. Furthermore, our transfer functions will be used to parametrize an uncertainty set, and as we shall see, it may be convenient to allow some of them to be only proper. A proportional-integral-derivative controller is very common in practice, especially in chemical engineering. It has the form

$$k_1 + \frac{k_2}{s} + k_3 s.$$

This is not proper, but it can be approximated over any desired frequency range by a proper one, for example,

$$k_1 + \frac{k_2}{s} + \frac{k_3 s}{\tau s + 1}.$$

Notice that the feedback system is automatically well-posed, in the stronger sense, if P, C, and F are proper and one is strictly proper. For most of the book, we shall make the following *standing assumption*, under which the nine transfer functions in (3.1) are proper:

P is strictly proper, C and F are proper.

However, at times it will be convenient to require only that P be proper. In this case we shall always assume that $|PCF| < 1$ at $\omega = \infty$, which ensures that $1 + PCF$ is not strictly proper. Given that no model, no matter how complex, can approximate a real system at sufficiently high frequencies, we should be very uncomfortable if $|PCF| > 1$ at $\omega = \infty$, because such a controller would almost surely be unstable if implemented on a real system.

3.2 Internal Stability

Consider a system with input u, output y, and transfer function $\hat{G}$, assumed stable and proper. We can write

$$\hat{G} = G_0 + \hat{G}_1,$$

where G_0 is a constant and $\hat{G}_1$ is strictly proper.

$$\text{Example:} \quad \frac{s}{s+1} = 1 - \frac{1}{s+1}.$$

In the time domain the equation is

$$y(t) = G_0 u(t) + \int_{-\infty}^{\infty} G_1(t - \tau) u(\tau)\, d\tau.$$

If $|u(t)| \le c$ for all t, then

$$|y(t)| \le |G_0| c + \int_{-\infty}^{\infty} |G_1(\tau)|\, d\tau c.$$

The right-hand side is finite. Thus the output is bounded whenever the input is bounded. [This argument is the basis for entry (2,2) in Table 2.2.]

If the nine transfer functions in (3.1) are stable, then the feedback system is said to be *internally stable*. As a consequence, if the exogenous inputs are bounded in magnitude, so too are x_1, x_2, and x_3, and hence u, y, and v. So internal stability guarantees bounded internal signals for all bounded exogenous signals.

The idea behind this definition of internal stability is that it is not enough to look only at input-output transfer functions, such as from r to y, for example. This transfer function could be stable, so that y is bounded when r is, and yet an internal signal could be unbounded, probably causing internal damage to the physical system.

For the remainder of this section hats are dropped.

Example In Figure 3.3 take

$$C(s) = \frac{s-1}{s+1}, \qquad P(s) = \frac{1}{s^2-1}, \qquad F(s) = 1.$$

Check that the transfer function from r to y is stable, but that from d to y is not. The feedback system is therefore not internally stable. As we will see later, this offense is caused by the cancellation of the controller zero and the plant pole at the point $s = 1$.

We shall develop a test for internal stability which is easier than examining nine transfer functions. Write P, C, and F as ratios of coprime polynomials (i.e., polynomials with no common factors):

$$P = \frac{N_P}{M_P}, \qquad C = \frac{N_C}{M_C}, \qquad F = \frac{N_F}{M_F}.$$

The *characteristic polynomial* of the feedback system is the one formed by taking the product of the three numerators plus the product of the three denominators:

$$N_P N_C N_F + M_P M_C M_F.$$

The *closed-loop poles* are the zeros of the characteristic polynomial.

Theorem 1 *The feedback system is internally stable iff there are no closed-loop poles in* $\mathrm{Re}\, s \geq 0$.

Proof For simplicity assume that $F = 1$; the proof in the general case is similar, but a bit messier.

From (3.1) we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{1+PC} \begin{bmatrix} 1 & -P & -1 \\ C & 1 & -C \\ PC & P & 1 \end{bmatrix} \begin{pmatrix} r \\ d \\ n \end{pmatrix}.$$

Substitute in the ratios and clear fractions to get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{N_P N_C + M_P M_C} \begin{bmatrix} M_P M_C & -N_P M_C & -M_P M_C \\ M_P N_C & M_P M_C & -M_P N_C \\ N_P N_C & N_P M_C & M_P M_C \end{bmatrix} \begin{pmatrix} r \\ d \\ n \end{pmatrix}. \tag{3.2}$$

Note that the characteristic polynomial equals $N_P N_C + M_P M_C$. Sufficiency is now evident; the feedback system is internally stable if the characteristic polynomial has no zeros in $\mathrm{Re}\, s \geq 0$.

Necessity involves a subtle point. Suppose that the feedback system is internally stable. Then all nine transfer functions in (3.2) are stable, that is, they have no poles in $\mathrm{Re}\ s \geq 0$. But we cannot immediately conclude that the polynomial $N_P N_C + M_P M_C$ has no zeros in $\mathrm{Re}\, s \geq 0$ because this polynomial may conceivably have a right half-plane zero which is also a zero of all nine numerators in (3.2), and hence is canceled to form nine stable transfer functions. However, the characteristic polynomial has no zero which is also a zero of all nine numerators, $M_P M_C$, $N_P M_C$, and so on.

Proof of this statement is left as an exercise. (It follows from the fact that we took coprime factors to start with, that is, N_P and M_P are coprime, as are the other numerator-denominator pairs.) ■

By Theorem 1 internal stability can be determined simply by checking the zeros of a polynomial. There is another test that provides additional insight.

Theorem 2 *The feedback system is internally stable iff the following two conditions hold:*

(a) The transfer function $1 + PCF$ has no zeros in Re$s \ge 0$.

(b) There is no pole-zero cancellation in Re$s \ge 0$ when the product PCF is formed.

Proof Recall that the feedback system is internally stable iff all nine transfer functions

$$\frac{1}{1+PCF}\begin{bmatrix} 1 & -PF & -F \\ C & 1 & -CF \\ PC & P & 1 \end{bmatrix}$$

are stable.

($\Rightarrow$) Assume that the feedback system is internally stable. Then in particular $(1 + PCF)^{-1}$ is stable (i.e., it has no poles in Re$s \ge 0$). Hence $1 + PCF$ has no zeros there. This proves (a).

To prove (b), write P, C, F as ratios of coprime polynomials:

$$P = \frac{N_P}{M_P}, \quad C = \frac{N_C}{M_C}, \quad F = \frac{N_F}{M_F}.$$

By Theorem 1 the characteristic polynomial

$$N_P N_C N_F + M_P M_C M_F$$

has no zeros in Re$s \ge 0$. Thus the pair (N_P, M_C) have no common zero in Re$s \ge 0$, and similarly for the other numerator-denominator pairs.

($\Leftarrow$) Assume (a) and (b). Factor P, C, F as above, and let s_0 be a zero of the characteristic polynomial, that is,

$$(N_P N_C N_F + M_P M_C M_F)(s_0) = 0.$$

We must show that Re$s_0 < 0$; this will prove internal stability by Theorem 1. Suppose to the contrary that Re$s_0 \ge 0$. If

$$(M_P M_C M_F)(s_0) = 0,$$

then

$$(N_P N_C N_F)(s_0) = 0.$$

But this violates (b). Thus

$$(M_P M_C M_F)(s_0) \ne 0,$$

so we can divide by it above to get

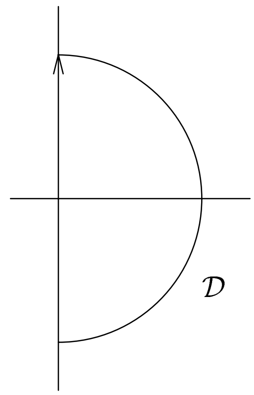
$$1 + \frac{N_P N_C N_F}{M_P M_C M_F}(s_0) = 0,$$

that is,

$$1 + (PCF)(s_0) = 0,$$

which violates (a). ■

Finally, let us recall for later use the Nyquist stability criterion. It can be derived from Theorem 2 and the principle of the argument. Begin with the curve $\mathcal{D}$ in the complex plane: It starts at the origin, goes up the imaginary axis, turns into the right half-plane following a semicircle of infinite radius, and comes up the negative imaginary axis to the origin again:



As a point s makes one circuit around this curve, the point $P(s)C(s)F(s)$ traces out a curve called the *Nyquist plot* of PCF. If PCF has a pole on the imaginary axis, then $\mathcal{D}$ must have a small indentation to avoid it.

Nyquist Criterion *Construct the Nyquist plot of PCF, indenting to the left around poles on the imaginary axis. Let n denote the total number of poles of P, C, and F in Res ≥ 0. Then the feedback system is internally stable iff the Nyquist plot does not pass through the point -1 and encircles it exactly n times counterclockwise.*

3.3 Asymptotic Tracking

In this section we look at a typical performance specification, perfect asymptotic tracking of a reference signal. Both time domain and frequency domain occur, so the notation distinction is required.

For the remainder of this chapter we specialize to the *unity-feedback* case, $\hat{F} = 1$, so the block diagram is as in Figure 3.4. Here e is the tracking error; with $n = d = 0$, e equals the reference input (ideal response), r, minus the plant output (actual response), y.

We wish to study this system's capability of tracking certain test inputs asymptotically as time tends to infinity. The two test inputs are the step

$$r(t) = \begin{cases} c, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

and the ramp

$$r(t) = \begin{cases} ct, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

(c is a nonzero real number). As an application of the former think of the temperature-control thermostat in a room; when you change the setting on the thermostat (step input), you would like

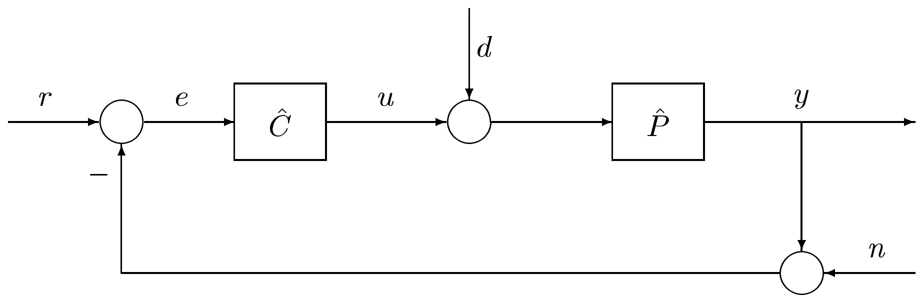


Figure 3.4: Unity-feedback loop.

the room temperature eventually to change to the new setting (of course, you would like the change to occur within a reasonable time). A situation with a ramp input is a radar dish designed to track orbiting satellites. A satellite moving in a circular orbit at constant angular velocity sweeps out an angle that is approximately a linear function of time (i.e., a ramp).

Define the *loop transfer function* $\hat{L} := \hat{P}\hat{C}$. The transfer function from reference input r to tracking error e is

$$\hat{S} := \frac{1}{1+\hat{L}},$$

called the *sensitivity function*—more on this in the next section. The ability of the system to track steps and ramps asymptotically depends on the number of zeros of $\hat{S}$ at $s = 0$.

Theorem 3 *Assume that the feedback system is internally stable and $n = d = 0$.*

(a) If r is a step, then $e(t) \longrightarrow 0$ as $t \longrightarrow \infty$ iff $\hat{S}$ has at least one zero at the origin.

(b) If r is a ramp, then $e(t) \longrightarrow 0$ as $t \longrightarrow \infty$ iff $\hat{S}$ has at least two zeros at the origin.

The proof is an application of the *final-value theorem*: If $\hat{y}(s)$ is a rational Laplace transform that has no poles in $\mathrm{Re}s \geq 0$ except possibly a simple pole at $s = 0$, then $\lim_{t\to\infty} y(t)$ exists and it equals $\lim_{s\to 0} s\hat{y}(s)$.

Proof (a) The Laplace transform of the foregoing step is $\hat{r}(s) = c/s$. The transfer function from r to e equals $\hat{S}$, so

$$\hat{e}(s) = \hat{S}(s)\frac{c}{s}.$$

Since the feedback system is internally stable, $\hat{S}$ is a stable transfer function. It follows from the final-value theorem that $e(t)$ does indeed converge as $t \longrightarrow \infty$, and its limit is the residue of the function $\hat{e}(s)$ at the pole $s = 0$:

$$e(\infty) = \hat{S}(0)c.$$

The right-hand side equals zero iff $\hat{S}(0) = 0$.

(b) Similarly with $\hat{r}(s) = c/s^2$. ■

Note that $\hat{S}$ has a zero at $s = 0$ iff $\hat{L}$ has a pole there. Thus, from the theorem we see that if the feedback system is internally stable and either $\hat{P}$ or $\hat{C}$ has a pole at the origin (i.e., an inherent integrator), then the output $y(t)$ will asymptotically track any step input r.

Example To see how this works, take the simplest possible example,

$$\hat{P}(s) = \frac{1}{s}, \quad \hat{C}(s) = 1.$$

Then the transfer function from r to e equals

$$\frac{1}{1+s^{-1}} = \frac{s}{s+1}.$$

So the open-loop pole at $s = 0$ becomes a closed-loop zero of the error transfer function; then this zero cancels the pole of $\hat{r}(s)$, resulting in no unstable poles in $\hat{e}(s)$. Similar remarks apply for a ramp input.

Theorem 3 is a special case of an elementary principle: For perfect asymptotic tracking, the loop transfer function $\hat{L}$ must contain an internal model of the unstable poles of $\hat{r}$.

A similar analysis can be done for the situation where $r = n = 0$ and d is a sinusoid, say $d(t) = \sin(\omega t)1(t)$ (1 is the unit step). You can show this: If the feedback system is internally stable, then $y(t) \longrightarrow 0$ as $t \longrightarrow \infty$ iff either $\hat{P}$ has a zero at $s = j\omega$ or $\hat{C}$ has a pole at $s = j\omega$ (Exercise 3).

3.4 Performance

In this section we again look at tracking a reference signal, but whereas in the preceding section we considered perfect asymptotic tracking of a *single* signal, we will now consider a *set* of reference signals and a bound on the steady-state error. This performance objective will be quantified in terms of a weighted norm bound.

As before, let L denote the loop transfer function, $L := PC$. The transfer function from reference input r to tracking error e is

$$S := \frac{1}{1+L},$$

called the *sensitivity function*. In the analysis to follow, it will always be assumed that the feedback system is internally stable, so S is a stable, proper transfer function. Observe that since L is strictly proper (since P is), $S(j\infty) = 1$.

The name *sensitivity function* comes from the following idea. Let T denote the transfer function from r to y:

$$T = \frac{PC}{1+PC}.$$

One way to quantify how sensitive T is to variations in P is to take the limiting ratio of a relative perturbation in T (i.e., $\Delta T/T$) to a relative perturbation in P (i.e., $\Delta P/P$). Thinking of P as a variable and T as a function of it, we get

$$\lim_{\Delta P \to 0} \frac{\Delta T/T}{\Delta P/P} = \frac{dT}{dP}\frac{P}{T}.$$

The right-hand side is easily evaluated to be S. In this way, S is the sensitivity of the closed-loop transfer function T to an infinitesimal perturbation in P.

Now we have to decide on a performance specification, a measure of goodness of tracking. This decision depends on two things: what we know about r and what measure we choose to assign to the tracking error. Usually, r is not known in advance—few control systems are designed for one and only one input. Rather, a set of possible rs will be known or at least postulated for the purpose of design.

Let's first consider sinusoidal inputs. Suppose that r can be any sinusoid of amplitude ≤ 1 and we want e to have amplitude $< \epsilon$. Then the performance specification can be expressed succinctly as

$$\|S\|_\infty < \epsilon.$$

Here we used Table 2.1: the maximum amplitude of e equals the ∞-norm of the transfer function. Or if we define the (trivial, in this case) weighting function $W_1(s) = 1/\epsilon$, then the performance specification is $\|W_1 S\|_\infty < 1$.

The situation becomes more realistic and more interesting with a frequency-dependent weighting function. Assume that $W_1(s)$ is real-rational; you will see below that only the magnitude of $W_1(j\omega)$ is relevant, so any poles or zeros in $\mathrm{Re}\, s > 0$ can be reflected into the left half-plane without changing the magnitude. Let us consider four scenarios giving rise to an ∞-norm bound on $W_1 S$. The first three require W_1 to be stable.

1. Suppose that the family of reference inputs is all signals of the form $r = W_1 r_{pf}$, where r_{pf}, a pre-filtered input, is any sinusoid of amplitude ≤ 1. Thus the set of rs consists of sinusoids with frequency-dependent amplitudes. Then the maximum amplitude of e equals $\|W_1 S\|_\infty$.

2. Recall from Chapter 2 that

$$\|r\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |r(j\omega)|^2 \, d\omega$$

and that $\|r\|_2^2$ is a measure of the energy of r. Thus we may think of $|r(j\omega)|^2$ as *energy spectral density*, or energy spectrum. Suppose that the set of all rs is

$$\{r : r = W_1 r_{pf}, \|r_{pf}\|_2 \leq 1\},$$

that is,

$$\left\{ r : \frac{1}{2\pi}\int_{-\infty}^{\infty} |r(j\omega)/W_1(j\omega)|^2 \, d\omega \leq 1 \right\}.$$

Thus, r has an energy constraint and its energy spectrum is weighted by $1/|W_1(j\omega)|^2$. For example, if W_1 were a bandpass filter, the energy spectrum of r would be confined to the passband. More generally, W_1 could be used to shape the energy spectrum of the expected class of reference inputs. Now suppose that the tracking error measure is the 2-norm of e. Then from Table 2.2,

$$\sup_r \|e\|_2 = \sup\{\|SW_1 r_{pf}\|_2 : \|r_{pf}\|_2 \leq 1\} = \|W_1 S\|_\infty,$$

so $\|W_1 S\|_\infty < 1$ means that $\|e\|_2 < 1$ for all rs in the set above .

3. This scenario is like the preceding one except for signals of finite power. We see from Table 2.2 that $\|W_1 S\|_\infty$ equals the supremum of $pow(e)$ over all r_{pf} with $pow(r_{pf}) \leq 1$. So W_1 could be used to shape the power spectrum of the expected class of rs.

4. In several applications, for example aircraft flight-control design, designers have acquired through experience desired shapes for the Bode magnitude plot of S. In particular, suppose that good performance is known to be achieved if the plot of $|S(j\omega)|$ lies under some curve. We could rewrite this as

$$|S(j\omega)| < |W_1(j\omega)|^{-1}, \quad \forall \omega,$$

or in other words, $\|W_1 S\|_\infty < 1$.

There is a nice graphical interpretation of the norm bound $\|W_1S\|_\infty < 1$. Note that

$$\begin{aligned} \|W_1S\|_\infty < 1 \quad &\Leftrightarrow \quad \left|\frac{W_1(j\omega)}{1+L(j\omega)}\right| < 1, \qquad \forall\omega \\ &\Leftrightarrow \quad |W_1(j\omega)| < |1+L(j\omega)|, \qquad \forall\omega. \end{aligned}$$

The last inequality says that at every frequency, the point $L(j\omega)$ on the Nyquist plot lies outside the disk of center -1, radius $|W_1(j\omega)|$ (Figure 3.5).

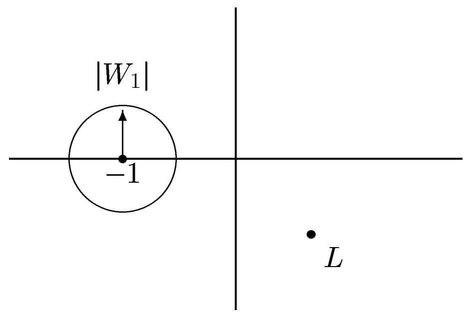


Figure 3.5: Performance specification graphically.

Other performance problems could be posed by focusing on the response to the other two exogenous inputs, d and n. Note that the transfer functions from d, n to e, u are given by

$$\begin{bmatrix} e \\ u \end{bmatrix} = -\begin{bmatrix} PS & S \\ T & CS \end{bmatrix} \begin{bmatrix} d \\ n \end{bmatrix},$$

where

$$T := 1 - S = \frac{PC}{1+PC},$$

called the *complementary sensitivity function*.

Various performance specifications could be made using weighted versions of the transfer functions above. Note that a performance spec with weight W on PS is equivalent to the weight WP on S. Similarly, a weight W on $CS = T/P$ is equivalent to the weight W/P on T. Thus performance specs that involve e result in weights on S and performance specs on u result in weights on T. Essentially all problems in this book boil down to weighting S or T or some combination, and the tradeoff between making S small and making T small is the main issue in design.

Exercises

1. Consider the unity-feedback system $[F(s) = 1]$. The definition of internal stability is that all nine closed-loop transfer functions should be stable. In the unity-feedback case, it actually suffices to check only two of the nine. Which two?

2. In this problem and the next, there is a mixture of the time and frequency domains, so the ˆ -convention is in force.

   Let

   $$\hat{P}(s) = \frac{1}{10s+1}, \qquad \hat{C}(s) = k, \qquad \hat{F}(s) = 1.$$

Find the least positive gain k so that the following are all true:

(a) The feedback system is internally stable.

(b) $|e(\infty)| \leq 0.1$ when $r(t)$ is the unit step and $n = d = 0$.

(c) $\|y\|_\infty \leq 0.1$ for all $d(t)$ such that $\|d\|_2 \leq 1$ when $r = n = 0$.

3. For the setup in Figure 3.4, take $r = n = 0$, $d(t) = \sin(\omega t)1(t)$. Prove that if the feedback system is internally stable, then $y(t) \to 0$ as $t \to \infty$ iff either $\hat{P}$ has a zero at $s = j\omega$ or $\hat{C}$ has a pole at $s = j\omega$.

4. Consider the feedback system with plant P and sensor F. Assume that P is strictly proper and F is proper. Find conditions on P and F for the existence of a proper controller so that

   The feedback system is internally stable.

   $y(t) - r(t) \to 0$ when r is a unit step.

   $y(t) \to 0$ when d is a sinusoid of frequency 100 rad/s.

Notes and References

The material in Sections 3.1 to 3.3 is quite standard. However, Section 3.4 reflects the more recent viewpoint of Zames (1981), who formulated the problem of optimizing W_1S with respect to the ∞-norm, stressing the role of the weight W_1. Additional motivation for this problem is offered in Zames and Francis (1983).

Chapter 5

Stabilization

In this chapter we study the unity-feedback system with block diagram shown in Figure 5.1. Here

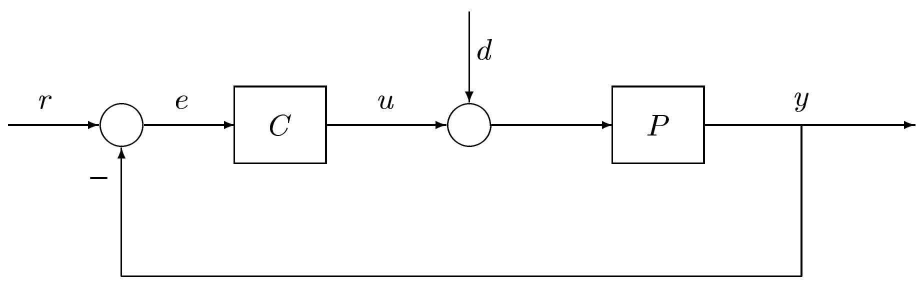


Figure 5.1: Unity-feedback system.

P is strictly proper and C is proper.

Most synthesis problems can be formulated in this way: Given P, design C so that the feedback system (1) is internally stable, and (2) acquires some additional desired property; for example, the output y asymptotically tracks a step input r. The method of solution is to parametrize all Cs for which (1) is true, and then to see if there exists a parameter for which (2) holds. In this chapter such a parametrization is derived and then applied to two problems: achieving asymptotic performance specs and internal stabilization by a stable controller.

5.1 Controller Parametrization: Stable Plant

In this section we assume that P is already stable, and we parametrize all Cs for which the feedback system is internally stable. Introduce the symbol $\mathcal{S}$ for the family of all stable, proper, real-rational functions. Notice that $\mathcal{S}$ is closed under addition and multiplication: If $F, G \in \mathcal{S}$, then $F+G, FG \in \mathcal{S}$. Also, $1 \in \mathcal{S}$. (Thus $\mathcal{S}$ is a commutative ring with identity.)

Theorem 1 *Assume that $P \in \mathcal{S}$. The set of all Cs for which the feedback system is internally stable equals*

$$\left\{ \frac{Q}{1-PQ} : Q \in \mathcal{S} \right\}.$$

Proof ($\subset$) Suppose that C achieves internal stability. Let Q denote the transfer function from r to u, that is,

$$Q := \frac{C}{1+PC}.$$

Then $Q \in \mathcal{S}$ and

$$C = \frac{Q}{1-PQ}.$$

($\supset$) Conversely, suppose that $Q \in \mathcal{S}$ and define

$$C := \frac{Q}{1-PQ}. \tag{5.1}$$

According to the definition in Section 3.2, the feedback system is internally stable iff the nine transfer functions

$$\frac{1}{1+PC}\begin{bmatrix} 1 & -P & -1 \\ C & 1 & -C \\ PC & P & 1 \end{bmatrix}$$

all are stable and proper. After substitution from (5.1) and clearing of fractions, this matrix becomes

$$\begin{bmatrix} 1-PQ & -P(1-PQ) & -(1-PQ) \\ Q & 1-PQ & -Q \\ PQ & P(1-PQ) & 1-PQ \end{bmatrix}.$$

Clearly, these nine entries belong to $\mathcal{S}$. ■

Note that all nine transfer functions above are affine functions of the free parameter Q; that is, each is of the form $T_1 + T_2 Q$ for some T_1, T_2 in $\mathcal{S}$. In particular the sensitivity and complementary sensitivity functions are

$$\begin{aligned} S &= 1 - PQ, \\ T &= PQ. \end{aligned}$$

Let us look at a simple application. Suppose that we want to find a C so that the feedback system is internally stable and y asymptotically tracks a step r (when $d = 0$). Parametrize C as in the theorem. Then y asymptotically tracks a step iff the transfer function from r to e (i.e., S) has a zero at $s = 0$, that is,

$$P(0)Q(0) = 1.$$

This equation has a solution Q in $\mathcal{S}$ iff $P(0) \neq 0$. Conclusion: The problem has a solution iff $P(0) \neq 0$; when this holds, the set of all solutions is

$$\left\{ C = \frac{Q}{1-PQ} : Q \in \mathcal{S}, Q(0) = \frac{1}{P(0)} \right\}.$$

Observe that Q inverts P at dc. Also, you can check that a controller of the latter form has a pole at $s = 0$, as it must by Theorem 3 of Chapter 3.

Example For the plant

$$P(s) = \frac{1}{(s+1)(s+2)}$$

suppose that it is desired to find an internally stabilizing controller so that y asymptotically tracks a ramp r. Parametrize C as in the theorem. The transfer function S from r to e must have (at least) two zeros at $s = 0$, where r has two poles. Let us take

$$Q(s) = \frac{as + b}{s + 1}.$$

This belongs to $\mathcal{S}$ and has two variables, a and b, for the assignment of the two zeros of S. We have

$$\begin{aligned} S(s) &= 1 - \frac{as + b}{(s + 1)^2(s + 2)} \\ &= \frac{s^3 + 4s^2 + (5 - a)s + (2 - b)}{(s + 1)^2(s + 2)}, \end{aligned}$$

so we should take $a = 5, b = 2$. This gives

$$\begin{aligned} Q(s) &= \frac{5s + 2}{s + 1}, \\ C(s) &= \frac{(5s + 2)(s + 1)(s + 2)}{s^2(s + 4)}. \end{aligned}$$

The controller is internally stabilizing and has two poles at $s = 0$.

5.2 Coprime Factorization

Now suppose that P is not stable and we want to find an internally stabilizing C. We might try as follows. Write P as the ratio of coprime polynomials,

$$P = \frac{N}{M}.$$

By Euclid's algorithm (reviewed below) we can get two other polynomials X, Y satisfying the equation

$$NX + MY = 1.$$

Remembering Theorem 3.1 (the feedback system is internally stable iff the characteristic polynomial has no zeros in Re $s \geq 0$), we might try to make the left-hand side equal to the characteristic polynomial by setting

$$C = \frac{X}{Y}.$$

The trouble is that Y may be 0; even if not, this C may not be proper.

Example 1 For $P(s) = 1/s$, we can take $N(s) = 1$, $M(s) = s$. One solution to the equation $NX + MY = 1$ is $X(s) = 1$, $Y(s) = 0$, for which X/Y is undefined. Another solution is $X(s) = -s + 1$, $Y(s) = 1$, for which X/Y is not proper.

The remedy is to arrange that N, M, X, Y are all elements of $\mathcal{S}$ instead of polynomials. Two functions N and M in $\mathcal{S}$ are *coprime* if there exist two other functions X and Y also in $\mathcal{S}$ and satisfying the equation

$$NX + MY = 1.$$

Notice that for this equation to hold, N and M can have no common zeros in $\mathrm{Re} s \geq 0$ nor at the point $s = \infty$—if there were such a point s_0, there would follow

$$0 = N(s_0)X(s_0) + M(s_0)Y(s_0) \neq 1.$$

It can be proved that this condition is also sufficient for coprimeness.

Let G be a real-rational transfer function. A representation of the form

$$G = \frac{N}{M}, \qquad N, M \in \mathcal{S},$$

where N and M are coprime, is called a *coprime factorization* of G over $\mathcal{S}$. The purpose of this section is to present a method for the construction of four functions in $\mathcal{S}$ satisfying the two equations

$$G = \frac{N}{M}, \qquad NX + MY = 1.$$

The construction of N and M is easy.

Example 2 Take $G(s) = 1/(s-1)$. To write $G = N/M$ with N and M in $\mathcal{S}$, simply divide the numerator and denominator polynomials, 1 and $s-1$, by a common polynomial with no zeros in $\mathrm{Re} s \geq 0$, say $(s+1)^k$:

$$\frac{1}{s-1} = \frac{N(s)}{M(s)}, \quad N(s) = \frac{1}{(s+1)^k}, \quad M(s) = \frac{s-1}{(s+1)^k}.$$

If the integer k is greater than 1, then N and M are not coprime—they have a common zero at $s = \infty$. So

$$N(s) = \frac{1}{s+1}, \quad M(s) = \frac{s-1}{s+1}$$

suffice.

More generally, to get N and M we could divide the numerator and denominator polynomials of G by $(s+1)^k$, where k equals the maximum of their degrees. What is not so easy is to get the other two functions, X and Y, and this is why we need Euclid's algorithm.

Euclid's algorithm computes the greatest common divisor of two given polynomials, say $n(\lambda)$ and $m(\lambda)$. When n and m are coprime, the algorithm can be used to compute polynomials $x(\lambda)$, $y(\lambda)$ satisfying

$$nx + my = 1.$$

Procedure A: Euclid's Algorithm

Input: polynomials n, m

Initialize: If it is not true that degree $(n) \geq$ degree (m), interchange n and m.

Step 1 Divide m into n to get quotient q_1 and remainder r_1:

$$n = mq_1 + r_1,$$

$$\text{degree } r_1 < \text{ degree } m.$$

Step 2 Divide r_1 into m to get quotient q_2 and remainder r_2:

$$m = r_1 q_2 + r_2,$$

$$\text{degree } r_2 < \text{ degree } r_1.$$

Step 3 Divide r_2 into r_1:

$$r_1 = r_2 q_3 + r_3,$$

$$\text{degree } r_3 < \text{ degree } r_2.$$

Continue.

Stop at Step k when r_k is a nonzero constant.

Then x, y are obtained as illustrated by the following example for $k = 3$. The equations are

$$n = m q_1 + r_1,$$

$$m = r_1 q_2 + r_2,$$

$$r_1 = r_2 q_3 + r_3,$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 \\ q_2 & 1 & 0 \\ -1 & q_3 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & -q_1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} n \\ m \end{bmatrix}.$$

Solve for r_3 by, say, Gaussian elimination:

$$r_3 = (1 + q_2 q_3) n + [-q_3 - q_1(1 + q_2 q_3)] m.$$

Set

$$\begin{aligned} x &= \frac{1}{r_3}(1 + q_2 q_3), \\ y &= \frac{1}{r_3}[-q_3 - q_1(1 + q_2 q_3)]. \end{aligned}$$

Example 3 The algorithm for $n(\lambda) = \lambda^2$, $m(\lambda) = 6\lambda^2 - 5\lambda + 1$ goes like this:

$$\begin{aligned} q_1(\lambda) &= \frac{1}{6}, \\ r_1(\lambda) &= \frac{5}{6}\lambda - \frac{1}{6}, \\ q_2(\lambda) &= \frac{36}{5}\lambda - \frac{114}{25}, \\ r_2(\lambda) &= \frac{6}{25}. \end{aligned}$$

Since r_2 is a nonzero constant, we stop after Step 2. Then the equations are

$$\begin{aligned} n &= m q_1 + r_1, \\ m &= r_1 q_2 + r_2, \end{aligned}$$

yielding

$$r_2 = (1 + q_1 q_2)m - q_2 n.$$

So we should take

$$x = -\frac{q_2}{r_2}, \quad y = \frac{1 + q_1 q_2}{r_2},$$

that is,

$$x(\lambda) = -30\lambda + 19, \quad y(\lambda) = 5\lambda + 1.$$

Next is a procedure for doing a coprime factorization of G. The main idea is to transform variables, $s \to \lambda$, so that polynomials in λ yield functions in $\mathcal{S}$.

Procedure B

Input: G

Step 1 If G is stable, set $N = G$, $M = 1$, $X = 0$, $Y = 1$, and stop; else, continue.

Step 2 Transform $G(s)$ to $\tilde{G}(\lambda)$ under the mapping $s = (1 - \lambda)/\lambda$. Write $\tilde{G}$ as a ratio of coprime polynomials:

$$\tilde{G}(\lambda) = \frac{n(\lambda)}{m(\lambda)}.$$

Step 3 Using Euclid's algorithm, find polynomials $x(\lambda)$, $y(\lambda)$ such that

$$nx + my = 1.$$

Step 4 Transform $n(\lambda)$, $m(\lambda)$, $x(\lambda)$, $y(\lambda)$ to $N(s)$, $M(s)$, $X(s)$, $Y(s)$ under the mapping $\lambda = 1/(s + 1)$.

The mapping used in this procedure is not unique; the only requirement is that polynomials n, and so on, map to N, and so on, in $\mathcal{S}$.

Example 4 For

$$G(s) = \frac{1}{(s-1)(s-2)}$$

the algorithm gives

$$\begin{aligned}
\tilde{G}(\lambda) &= \frac{\lambda^2}{6\lambda^2 - 5\lambda + 1}, \\
n(\lambda) &= \lambda^2, \\
m(\lambda) &= 6\lambda^2 - 5\lambda + 1, \\
x(\lambda) &= -30\lambda + 19, \\
y(\lambda) &= 5\lambda + 1 \quad \text{(from Example 3)}, \\
N(s) &= \frac{1}{(s+1)^2},
\end{aligned}$$

$$\begin{aligned} M(s) &= \frac{(s-1)(s-2)}{(s+1)^2}, \\ X(s) &= \frac{19s-11}{s+1}, \\ Y(s) &= \frac{s+6}{s+1}. \end{aligned}$$

5.3 Coprime Factorization by State-Space Methods (Optional)

This optional section presents a state-space procedure for computing a coprime factorization over $\mathcal{S}$ of a proper G. This procedure is more efficient than the polynomial method in the preceding section.

We start with a new data structure. Suppose that A, B, C, D are real matrices of dimensions

$$n \times n, \quad n \times 1, \quad 1 \times n, \quad 1 \times 1.$$

The transfer function going along with this quartet is

$$D + C(sI - A)^{-1}B.$$

Note that the constant D equals the value of the transfer function at $s = \infty$; if the transfer function is strictly proper, then $D = 0$. It is convenient to write

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

instead of

$$D + C(sI - A)^{-1}B.$$

Beginning with a realization of G,

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

the goal is to get state-space realizations for four functions N, M, X, Y, all in $\mathcal{S}$, such that

$$G = \frac{N}{M}, \quad NX + MY = 1.$$

First, we look at how to get N and M. If the input and output of G are denoted u and y, respectively, then the state model of G is

$$\dot{x} = Ax + Bu, \tag{5.2}$$
$$y = Cx + Du. \tag{5.3}$$

Choose a real matrix F, $1 \times n$, such that $A + BF$ is stable (i.e., all eigenvalues in $\mathrm{Re}\, s < 0$). Now define the signal $v := u - Fx$. Then from (5.2) and (5.3) we get

$$\begin{aligned} \dot{x} &= (A + BF)x + Bv, \\ u &= Fx + v, \\ y &= (C + DF)x + Dv. \end{aligned}$$

Evidently from these equations, the transfer function from v to u is

$$M(s) := \left[\begin{array}{c|c} A+BF & B \\ \hline F & 1 \end{array}\right], \tag{5.4}$$

and that from v to y is

$$N(s) := \left[\begin{array}{c|c} A+BF & B \\ \hline C+DF & D \end{array}\right]. \tag{5.5}$$

Therefore,

$$u = Mv, \quad y = Nv,$$

so that $y = NM^{-1}u$, that is, $G = N/M$. Clearly, N and M are proper, and they are stable because $A+BF$ is. Thus $N, M \in \mathcal{S}$. Suggestion: Test the formulas above for the simplest case, $G(s) = 1/s$ ($A = 0$, $B = 1$, $C = 1$, $D = 0$).

The theory behind the formulas for X and Y is beyond the scope of this book. The procedure is to choose a real matrix H, $n \times 1$, so that $A + HC$ is stable, and then set

$$X(s) := \left[\begin{array}{c|c} A+HC & H \\ \hline F & 0 \end{array}\right], \tag{5.6}$$

$$Y(s) := \left[\begin{array}{c|c} A+HC & -B-HD \\ \hline F & 1 \end{array}\right]. \tag{5.7}$$

In summary, the procedure to do a coprime factorization of G is this:

Step 1 Get a realization (A, B, C, D) of G.

Step 2 Compute matrices F and H so that $A + BF$ and $A + HC$ are stable.

Step 3 Using formulas (5.4) to (5.7), compute the four functions N, M, X, Y.

5.4 Controller Parametrization: General Plant

The transfer function P is no longer assumed to be stable. Let $P = N/M$ be a coprime factorization over $\mathcal{S}$ and let X, Y be two functions in $\mathcal{S}$ satisfying the equation

$$NX + MY = 1. \tag{5.8}$$

Theorem 2 *The set of all Cs for which the feedback system is internally stable equals*

$$\left\{ \frac{X + MQ}{Y - NQ} : Q \in \mathcal{S} \right\}.$$

It is useful to note that Theorem 2 reduces to Theorem 1 when $P \in \mathcal{S}$. To see this, recall from Section 5.2 (Step 1 of Procedure B) that we can take

$$N = P, \quad M = 1, \quad X = 0, \quad Y = 1$$

when $P \in \mathcal{S}$. Then

$$\frac{X + MQ}{Y - NQ} = \frac{Q}{1 - PQ}.$$

The proof of Theorem 2 requires a preliminary result.

Lemma 1 *Let $C = N_C/M_C$ be a coprime factorization over $\mathcal{S}$. Then the feedback system is internally stable iff*

$$(NN_C + MM_C)^{-1} \in \mathcal{S}.$$

The proof of this lemma is almost identical to the proof of Theorem 3.1, and so is omitted.

Proof of Theorem 2 ($\supset$) Suppose that $Q \in \mathcal{S}$ and

$$C := \frac{X + MQ}{Y - NQ}.$$

To show that the feedback system is internally stable, define

$$N_C := X + MQ, \quad M_C := Y - NQ.$$

Then from the equation

$$NX + MY = 1$$

it follows that

$$NN_C + MM_C = 1.$$

Therefore, $C = N_C/M_C$ is a coprime factorization, and from Lemma 1 the feedback system is internally stable.

($\subset$) Conversely, let C be any controller achieving internal stability. We must find a Q in $\mathcal{S}$ such that

$$C = \frac{X + MQ}{Y - NQ}.$$

Let $C = N_C/M_C$ be a coprime factorization over $\mathcal{S}$ and define

$$V := (NN_C + MM_C)^{-1}$$

so that

$$NN_CV + MM_CV = 1. \tag{5.9}$$

By Lemma 1, $V \in \mathcal{S}$. Let Q be the solution of

$$M_CV = Y - NQ. \tag{5.10}$$

Substitute (5.10) into (5.9) to get

$$NN_CV + M(Y - NQ) = 1. \tag{5.11}$$

Also, add and subtract NMQ in (5.8) to give

$$N(X + MQ) + M(Y - NQ) = 1. \tag{5.12}$$

Comparing (5.11) and (5.12), we see that

$$N_CV = X + MQ. \tag{5.13}$$

Now (5.10) and (5.13) give

$$C = \frac{N_CV}{M_CV} = \frac{X + MQ}{Y - NQ}.$$

It remains to show that $Q \in \mathcal{S}$. Multiply (5.10) by X and (5.13) by Y, then subtract and switch sides:

$$(NX + MY)Q = YN_CV - XM_CV.$$

But the left-hand side equals Q by (5.8), while the right-hand side belongs to $\mathcal{S}$. So we are done. ■

Theorem 2 gives an automatic way to stabilize a plant.

Example Let

$$P(s) = \frac{1}{(s-1)(s-2)}.$$

Apply Procedure B to get

$$\begin{aligned}
N(s) &= \frac{1}{(s+1)^2}, \\
M(s) &= \frac{(s-1)(s-2)}{(s+1)^2}, \\
X(s) &= \frac{19s-11}{s+1}, \\
Y(s) &= \frac{s+6}{s+1}.
\end{aligned}$$

According to the theorem, the controller

$$C(s) = \frac{X(s)}{Y(s)} = \frac{19s-11}{s+6}$$

achieves internal stability.

As before, when P was stable, all closed-loop transfer functions are affine functions of Q if C is parametrized as in the theorem statement. For example, the sensitivity and complementary sensitivity functions are

$$\begin{aligned}
S &= M(Y - NQ), \\
T &= N(X + MQ).
\end{aligned}$$

Finally, it is sometimes useful to note that Lemma 1 suggests another way to solve the equation $NX + MY = 1$ given coprime N and M. First, find a controller C achieving internal stability for $P = N/M$—this might be easier than solving for X and Y. Next, write a coprime factorization of C: $C = N_C/M_C$. Then Lemma 1 says that

$$V := NN_C + MM_C$$

is invertible in $\mathcal{S}$. Finally, set $X = N_CV^{-1}$ and $Y = M_CV^{-1}$.

5.5 Asymptotic Properties

How to find a C to achieve internal stability and asymptotic properties simultaneously is perhaps best shown by an example.

Let

$$P(s) = \frac{1}{(s-1)(s-2)}.$$

The problem is to find a proper C so that