Robust Control

Lecture 1: introduction, norms and spaces.

Carolina Bergeling

Robust Control, 9hp

- 7 Lectures, 7 exercises
- Literature: Essentials of Robust Control, Zhou/Doyle, + Handouts
- Tools: Matlab
- Schedule and material: see Canvas page
- Examination: Exercises + Handins + Exam
- Collaboration encouraged on exercises and handins!
- Handins are due before the exercise session, email to: carolina.bergeling@control.lth.se with subject *Robust control handin X*
- Prepare so that you are able to share your solutions to the exercises at the session. (Take a photo of handwritten notes or typeset)

Contents

Controls education (related to robust control)

- Linear Algebra
- Control course, basic
- Matrix Theory
- Multivariable Control
- Functional Analysis (for Systems Theory)
- Linear Systems
- Robust Control

Lecture 1 - today

- Why robust control?
- What the course (and book) is about
- How to compare systems - Norms and spaces

Why robust control?

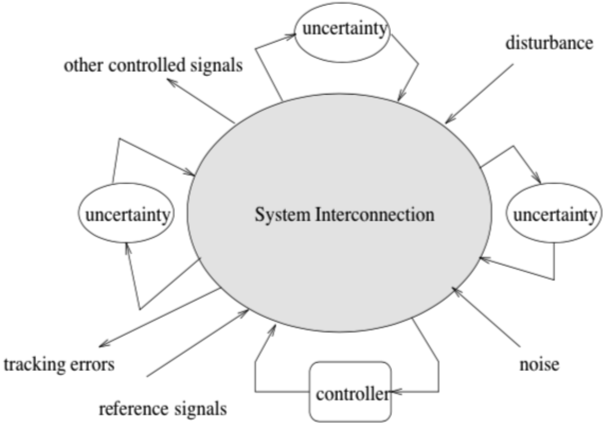

Figure 1.1: General system interconnection

Doyle's counterexample

Background: LQR guarantees 60° phase margin and 6 dB gain margin. Does there exist similar guarantees for LQG (Kalman filter in the loop)?

Doyle's counterexample

Counterexample: Given

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v,$$

with $Q = qCC^T$, $q > 0, R = 1$, the optimal control and filter gain vectors are given by

$$L = f \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad K = d \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $f = 2 + \sqrt{4 + q}$ and $d = 2 + \sqrt{4 + \sigma}$.

Doyle's counterexample

Assume $u = -mLx$, m nominally equals 1. Then stability (as dependent on m) requires

$$d + f - 4 + 2(m-1)df > 0$$
$$1 + (1-m)df > 0$$

Doyle's counterexample

Assume $u = -mLx$, m nominally equals 1. Then stability (as dependent on m) requires

$$d + f - 4 + 2(m-1)df > 0$$
$$1 + (1-m)df > 0$$

For sufficiently large f and d (or q and σ), the system is unstable for arbitrarily small perturbations in m in either direction.

What is this course about?

We design a controller C for a mathematical model M and want the corresponding real process P to behave well.

Problems:

- $P \neq M$
- Even if $P = M$ there is controller implementation errors

Robustness philosophy: The controller C is *robust* if

$$\begin{array}{rcl} P & \approx & M \\ C_r & \approx & C \end{array} \quad \Rightarrow \quad (P, C_r) \approx (M, C).$$

- What does it mean "≈"? (This lecture)
- How to check this? — Analysis.
- How to find the controller? — Synthesis

What does “$\approx$” mean?

We want to be able to compare different systems. How to do that?

What does "$\approx$" mean?

We want to be able to compare different systems. How to do that?

- Simpler question, how do you compare the maginitude of a scalar x with a scalar y?

What does "≈" mean?

We want to be able to compare different systems. How to do that?

- Simpler question, how do you compare the maginitude of a scalar x with a scalar y?
- One level up, what if x and y are vectors in R^n?

What does "$\approx$" mean?

We want to be able to compare different systems. How to do that?

- Simpler question, how do you compare the maginitude of a scalar x with a scalar y?
- One level up, what if x and y are vectors in R^n?
- Yet another level up, what if x and y are functions of time, i.e., $x(t)$ and $y(t)$?

What does “$\approx$” mean?

We want to be able to compare different systems. How to do that?

- Simpler question, how do you compare the maginitude of a scalar x with a scalar y?
- One level up, what if x and y are vectors in R^n?
- Yet another level up, what if x and y are functions of time, i.e., $x(t)$ and $y(t)$?
- And what if we want to compare a system G_1 to another system G_2?

What does "≈" mean?

We want to be able to compare different systems. How to do that?

- Simpler question, how do you compare the maginitude of a scalar x with a scalar y?
- One level up, what if x and y are vectors in R^n?
- Yet another level up, what if x and y are functions of time, i.e., $x(t)$ and $y(t)$?
- And what if we want to compare a system G_1 to another system G_2?

Dream: To use intuition from $\mathbb{R}^n$ in more general situations

Linear (or vector) space

Consider a set $X = \{x\}$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with two operations $+ : X \times X \to X$ and $\cdot : \mathbb{F} \times X \to X$. Then X is a linear space if

1. $x_1 + x_2 = x_2 + x_1$.
2. $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$.
3. $\exists 0 \in X$ such that $x + 0 = x \ \forall x \in X$.
4. $\forall x \in X \ \exists (-x) \in X$ such that $x + (-x) = 0$.
5. $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$.
6. $\lambda(x_1 + x_2) = \lambda x_1 + \lambda x_2$.
7. $\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2)x$.
8. $1x = x$.

Linear (or vector) space

Consider a set $X = \{x\}$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with two operations $+: X \times X \to X$ and $\cdot: \mathbb{F} \times X \to X$.
Then X is a linear space if

1. $x_1 + x_2 = x_2 + x_1$.
2. $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$.
3. $\exists 0 \in X$ such that $x + 0 = x \; \forall x \in X$.
4. $\forall x \in X \; \exists (-x) \in X$ such that $x + (-x) = 0$.
5. $(\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x$.
6. $\lambda(x_1 + x_2) = \lambda x_1 + \lambda x_2$.
7. $\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2)x$.
8. $1x = x$.

Example 1: R^n or C^n

Example 2: functions from any field Ω to $\mathbb{F}$
$(f + g)(t) = f(t) + g(t)$
$(\lambda \cdot g)(t) = \lambda \cdot g(t)$

The space of linear systems

Denote by $\mathscr{L}$ the set of all linear systems. It becomes the linear space with the following natural definition of $+$ and $\cdot$

$$\begin{array}{rcl} y_1 & = & G_1 u, \\ y_2 & = & G_2 u \end{array} \Rightarrow (G_1 + G_2)u = y_1 + y_2,$$

$$y = Gu \Rightarrow (\lambda G)u = \lambda y.$$

The space of linear systems

Denote by $\mathscr{L}$ the set of all linear systems. It becomes the linear space with the following natural definition of + and ·

$$\begin{array}{rcl} y_1 & = & G_1 u, \\ y_2 & = & G_2 u \end{array} \Rightarrow (G_1 + G_2)u = y_1 + y_2,$$

$$y = Gu \Rightarrow (\lambda G)u = \lambda y.$$

Only algebraic linearity is rather poor generalization of $\mathbb{R}^n$. What about the distance between two linear systems? What does it mean

$$G_1 \approx G_2?$$

Normed linear space

A linear space X is called *normed* if every vector $x \in X$ has an associated real number $\|x\|$ — its “length”, called the norm of the vector x, — with the following properties

1. $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.
2. $\|\lambda x\| = |\lambda| \|x\|$.
3. $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$.

Now we can say that $x_1 \approx x_2$ if $\|x_2 - x_1\|$ is small.

Normed linear space

A linear space X is called *normed* if every vector $x \in X$ has an associated real number $\|x\|$ — its “length”, called the norm of the vector x, — with the following properties

1. $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.
2. $\|\lambda x\| = |\lambda| \|x\|$.
3. $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$.

Now we can say that $x_1 \approx x_2$ if $\|x_2 - x_1\|$ is small.

Norms for signals

Consider signals mapping $(-\infty, \infty)$ to $\mathbb{R}$ (piecewise continuous)

Some norms for a signal $u(t)$

$$\|u\|_1 = \int_{-\infty}^{\infty} |u(t)| dt$$

$$\|u\|_2 = \left(\int_{-\infty}^{\infty} u(t)^2 dt \right)^{\frac{1}{2}}$$

$$\|u\|_\infty = \sup_t |u(t)|$$

Norms for signals

Consider signals mapping $(-\infty, \infty)$ to $\mathbb{R}$ (piecewise continuous)

Some norms for a signal $u(t)$

$$\|u\|_1 = \int_{-\infty}^{\infty} |u(t)| dt$$

$$\|u\|_2 = \left(\int_{-\infty}^{\infty} u(t)^2 dt \right)^{\frac{1}{2}}$$

$$\|u\|_\infty = \sup_t |u(t)|$$

Reminder: For $u \in \mathbb{R}^N$

$$\|u\|_1 = \sum_{i=1}^{N} |u_i|$$

$$\|u\|_2 = \left(\sum_{i=1}^{N} u_i^2 \right)^{\frac{1}{2}}$$

$$\|u\|_\infty = \max_i |u_i|$$

System definition and properties

Consider systems that are linear, time-invariant, causal, and finite-dimensional.

Time domain: $y = g * u = \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau$. (Causality means that $g(t) = 0$ for $t < 0$.)

Frequency domain: $\hat{y} = G\hat{u}$ where G is the Laplace transform of g

G is

- rational by finite-dimensionality, and has real coefficients.
- *stable* if it is analytic in the closed right half-plane
- *proper* if $G(j\infty)$ is finite
- *strictly proper* if $G(j\infty) = 0$

Norms for systems

Some norms for the transfer function G

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega \right)^{\frac{1}{2}}$$

$$\|G\|_\infty = \sup_\omega |G(j\omega)|$$

Notice that if G is stable, then $\|G\|_2 = \|g\|_2$ (by Parseval's theorem).

When are they finite? No poles on imaginary axis, and strictly proper/proper for 2-norm and ∞-norm, respectively.

Input-Output Relationships

G stable and strictly proper.

	$u(t) = \delta(t)$	$u(t) = \sin(\omega t)$
$\|y\|_2$	$\|G\|_2$	∞
$\|y\|_\infty$	$\|g\|_\infty$	$\lvert G(j\omega)\rvert$

Input-Output Relationships

G stable and strictly proper.

	$\|u\|_2$	$\|u\|_\infty$
$\|y\|_2$	$\|G\|_\infty$	∞
$\|y\|_\infty$	$\|G\|_2$	$\|g\|_1$

Entries given by $\sup_{\|u\|_U \leq 1} \|y\|_Y$ - what is such a norm called?

Induced norm

A linear system can be considered as an operator from the input space U to the output space Y. If U and Y are normed linear spaces then the following system norm is said to be *induced* by the signal norms on U and Y

$$\|G\| = \sup_{\|u\|_U \leq 1} \|Gu\|_Y.$$

Now we can compare G_1 and G_2 through $\|G_1 - G_2\|$.

Banach spaces

A complete normed linear space is called Banach space.

Completeness means that there are no holes in the space. (Cauchy sequences converge to a well defined limit within the space)

Hilbert spaces

An *inner product* is a functional $\langle , \rangle$ with the properties

1. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$.
2. $\langle x_1, x_2 \rangle = \overline{\langle x_2, x_1 \rangle}$.
3. $\langle x_1 + x_2, x_3 \rangle = \langle x_1, x_3 \rangle + \langle x_2, x_3 \rangle$.
4. $\langle \lambda x_1, x_2 \rangle = \lambda \langle x_1, x_2 \rangle$.

If there is an inner product on X then the norm can be defined as

$$\|x\| = \sqrt{\langle x, x \rangle}. \tag{1}$$

A Banach space with inner product and the norm (1) is called Hilbert space.

Remark:

- Existence of the inner product gives an additional nice property of the corresponding norm which makes the space be very similar to $\mathbb{R}^n$. This property is

$$\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 = 2(\|x_1\|^2 + \|x_2\|^2).$$

It simplifies drastically the optimization in Hilbert spaces.

Examples: L_2 and L_∞ spaces.

Example 1: L_2 space. Consider the linear space of all matrix-valued functions on $\mathbb{R}$

$$L_2(\mathbb{R}) = \{F \,:\, \int_{\mathbb{R}} \mathrm{tr}[F(t)^* F(t)]\, dt < +\infty\}.$$

This is the Hilbert space with the inner product

$$\langle F, G \rangle_2 = \int_{\mathbb{R}} \mathrm{tr}[F(t)^* G(t)]\, dt$$

Example 2: L_∞ space. Consider the linear space of all matrix-valued functions on $\mathbb{R}$

$$L_\infty(\mathbb{R}) = \{F \,:\, \text{ess sup}\, \sigma_{max}[F(t)] < +\infty\}.$$

This is a Banach space with $\|F\|_\infty = \text{ess sup}_{t\in\mathbb{R}}\, \sigma_{\max}[F(t)]$

Choice of U and Y as L_2.

One of the simplest choices of the input and output spaces is L_2 mainly because it is the Hilbert space. In this case the linear system G is a stable linear operator on $L_2[0,\infty)$

$$G\colon L_2[0,\infty) \to L_2[0,\infty)$$

and the norm of the linear system is L_2-induced norm

$$\|G\| = \sup_{\|u\|_2 \le 1} \|Gu\|_2 = \|G(j\omega)\|_\infty$$

where $G(s)$ is the transfer function of LTI system (Parseval’s relation + Theorem 4.3 in [Zhou+Doyle]).

Stability and Hardy spaces.

Stability is a very important issue in system analysis.

This motivates the introduction of *Hardy spaces*:

Define for $p = 2$ and $p = \infty$

$$\begin{aligned} H_p &= \{F \in L_p(j\mathbb{R}) : \ F \text{ is analytic in the right half plane}\} \\ \|F\|_{H_p} &= \sup_{\sigma > 0} \|F(\sigma + j\omega)\|_{L_p}. \end{aligned}$$

Are these norms easy to compute?

If G is stable, rational and strictly proper, then

$$\|G\|_p := \|G(j\omega)\|_{L_p} = \|G\|_{H_p}.$$

Notice that $\|G\|_2$ is finite if only if G is strictly proper.

L_2/H_2 **norm:**

Theorem 1: Let $G(s) = C(sI - A)^{-1}B$ and A is stable matrix. Then

$$\|G\|_2^2 = \mathrm{tr}(B^* Q B) = \mathrm{tr}(C P C^*)$$

where P is controllability and Q is observability Gramian

$$\begin{aligned} AP + PA^* + BB^* &= 0, \\ A^*Q + QA + C^*C &= 0. \end{aligned}$$

The formula for $\|G\|_2$

The transfer function $G(s)$ is the Laplace transform of the impulse response

$$g(t) = \begin{cases} Ce^{At}B, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Hence by Parseval's formula

$$\begin{aligned} \|G\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\{G(i\omega)^* G(i\omega)\} d\omega = \int_0^{\infty} \mathrm{tr}\{g(t)^* g(t)\} dt \\ &= \int_0^{\infty} \mathrm{tr}\{B^* e^{A^* t} C^* C e^{At} B\} dt = \mathrm{tr}(B^* Q B) \end{aligned}$$

since

$$Q = \int_0^{\infty} e^{A^* t} C^* C e^{At} dt$$

L_∞/H_∞ norm:

For real-rational plants $\|G\|_\infty < +\infty$ only if $G(s)$ is proper.

The computation is more complicated than for H_2 norm and requires a search.

Theorem 2: Let $G(s) = C(sI - A)^{-1}B + D \in H_\infty$. Then $\|G\|_\infty < \gamma$ if and only if

1. $\sigma_{\max}(D) < \gamma$,
2. H has no eigenvalues on the imaginary axis

where $R = \gamma^2 I - D^* D$ and

$$H = \begin{pmatrix} A+BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I+DR^{-1}D^*)C & -(A+BR^{-1}D^*C)^* \end{pmatrix}$$

What have we learned today?

- Robustness as a property of the closed-loop system to have similar behavior for all plants “close” to the nominal one.
- Normed linear space as the main tool to handle “close-far” notion. G_1 is “close” to G_2 $\leftrightarrow \|G_1 - G_2\|$ is small.
- $\|G\|$ depends on norms of input and output signal spaces.
- L_2 and L_∞ plus stability gives H_2 and H_∞. These are the most important spaces in the theory of robust control.
- They are also not very hard to compute — H_2 easier, H_∞ harder (needs an iteration).