Chapter 2

Norms for Signals and Systems

One way to describe the performance of a control system is in terms of the size of certain signals of interest. For example, the performance of a tracking system could be measured by the size of the error signal. This chapter looks at several ways of defining a signal's size (i.e., at several norms for signals). Which norm is appropriate depends on the situation at hand. Also introduced are norms for a system's transfer function. Then two very useful tables are developed summarizing input-output norm relationships.

2.1 Norms for Signals

We consider signals mapping $(-\infty, \infty)$ to $\mathbb{R}$. They are assumed to be piecewise continuous. Of course, a signal may be zero for $t < 0$ (i.e., it may start at time $t = 0$).

We are going to introduce several different norms for such signals. First, recall that a norm must have the following four properties:

(i) $\|u\| \geq 0$

(ii) $\|u\| = 0 \Leftrightarrow u(t) = 0, \qquad \forall t$

(iii) $\|au\| = |a|\|u\|, \qquad \forall a \in \mathbb{R}$

(iv) $\|u + v\| \leq \|u\| + \|v\|$

The last property is the familiar triangle inequality.

1-Norm The 1-norm of a signal $u(t)$ is the integral of its absolute value:

$$\|u\|_1 := \int_{-\infty}^{\infty} |u(t)| dt.$$

2-Norm The 2-norm of $u(t)$ is

$$\|u\|_2 := \left(\int_{-\infty}^{\infty} u(t)^2 dt \right)^{1/2}.$$

For example, suppose that u is the current through a 1 Ω resistor. Then the instantaneous power equals $u(t)^2$ and the total energy equals the integral of this, namely, $\|u\|_2^2$. We shall generalize this interpretation: The *instantaneous power* of a signal $u(t)$ is defined to be $u(t)^2$ and its *energy* is defined to be the square of its 2-norm.

∞-Norm The ∞-norm of a signal is the least upper bound of its absolute value:

$$\|u\|_\infty := \sup_t |u(t)|.$$

For example, the ∞-norm of

$$(1 - \mathrm{e}^{-t})1(t)$$

equals 1. Here $1(t)$ denotes the unit step function.

Power Signals The *average power* of u is the average over time of its instantaneous power:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)^2 dt.$$

The signal u will be called a *power signal* if this limit exists, and then the squareroot of the average power will be denoted $pow(u)$:

$$pow(u) := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)^2 dt \right)^{1/2}.$$

Note that a nonzero signal can have zero average power, so pow is not a norm. It does, however, have properties (i), (iii), and (iv).

Now we ask the question: Does finiteness of one norm imply finiteness of any others? There are some easy answers:

1. If $\|u\|_2 < \infty$, then u is a power signal with $pow(u) = 0$.

   Proof Assuming that u has finite 2-norm, we get

   $$\frac{1}{2T} \int_{-T}^{T} u(t)^2 dt \le \frac{1}{2T} \|u\|_2^2.$$

   But the right-hand side tends to zero as $T \to \infty$. ■

2. If u is a power signal and $\|u\|_\infty < \infty$, then $pow(u) \le \|u\|_\infty$.

   Proof We have

   $$\frac{1}{2T} \int_{-T}^{T} u(t)^2 dt \le \|u\|_\infty^2 \frac{1}{2T} \int_{-T}^{T} dt = \|u\|_\infty^2.$$

   Let T tend to ∞. ■

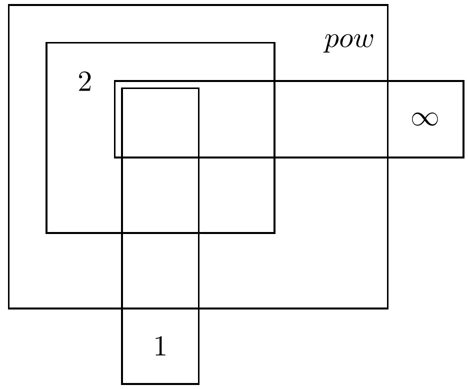

Figure 2.1: Set inclusions.

3. If $\|u\|_1 < \infty$ and $\|u\|_\infty < \infty$, then $\|u\|_2 \leq (\|u\|_\infty \|u\|_1)^{1/2}$, and hence $\|u\|_2 < \infty$.

   Proof

$$\int_{-\infty}^{\infty} u(t)^2 dt = \int_{-\infty}^{\infty} |u(t)||u(t)| dt \leq \|u\|_\infty \|u\|_1 \blacksquare$$

A Venn diagram summarizing the set inclusions is shown in Figure 2.1. Note that the set labeled "*pow*" contains all power signals for which *pow* is finite; the set labeled "1" contains all signals of finite 1-norm; and so on. It is instructive to get examples of functions in all the components of this diagram (Exercise 2). For example, consider

$$u_1(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1/\sqrt{t}, & \text{if } 0 < t \leq 1 \\ 0, & \text{if } t > 1. \end{cases}$$

This has finite 1-norm:

$$\|u_1\|_1 = \int_0^1 \frac{1}{\sqrt{t}} dt = 2.$$

Its 2-norm is infinite because the integral of $1/t$ is divergent over the interval $[0, 1]$. For the same reason, u_1 is not a power signal. Finally, u_1 is not bounded, so $\|u_1\|_\infty$ is infinite. Therefore, u_1 lives in the bottom component in the diagram.

2.2 Norms for Systems

We consider systems that are linear, time-invariant, causal, and (usually) finite-dimensional. In the time domain an input-output model for such a system has the form of a convolution equation,

$$y = G * u,$$

that is,

$$y(t) = \int_{-\infty}^{\infty} G(t-\tau)u(\tau)d\tau.$$

Causality means that $G(t) = 0$ for $t < 0$. Let $\hat{G}(s)$ denote the transfer function, the Laplace transform of G. Then $\hat{G}$ is rational (by finite-dimensionality) with real coefficients. We say that $\hat{G}$ is *stable* if it is analytic in the closed right half-plane (Re $s \geq 0$), *proper* if $\hat{G}(j\infty)$ is finite (degree of denominator $\geq$ degree of numerator), *strictly proper* if $\hat{G}(j\infty) = 0$ (degree of denominator $>$ degree of numerator), and *biproper* if $\hat{G}$ and $\hat{G}^{-1}$ are both proper (degree of denominator = degree of numerator).

We introduce two norms for the transfer function $\hat{G}$.

2-Norm

$$\|\hat{G}\|_2 := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 d\omega \right)^{1/2}$$

∞-Norm

$$\|\hat{G}\|_\infty := \sup_\omega |\hat{G}(j\omega)|$$

Note that if $\hat{G}$ is stable, then by Parseval's theorem

$$\|\hat{G}\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 d\omega \right)^{1/2} = \left(\int_{-\infty}^{\infty} |G(t)|^2 dt \right)^{1/2}.$$

The ∞-norm of $\hat{G}$ equals the distance in the complex plane from the origin to the farthest point on the Nyquist plot of $\hat{G}$. It also appears as the peak value on the Bode magnitude plot of $\hat{G}$. An important property of the ∞-norm is that it is submultiplicative:

$$\|\hat{G}\hat{H}\|_\infty \leq \|\hat{G}\|_\infty \|\hat{H}\|_\infty.$$

It is easy to tell when these two norms are finite.

Lemma 1 *The 2-norm of $\hat{G}$ is finite iff $\hat{G}$ is strictly proper and has no poles on the imaginary axis; the ∞-norm is finite iff $\hat{G}$ is proper and has no poles on the imaginary axis.*

Proof Assume that $\hat{G}$ is strictly proper, with no poles on the imaginary axis. Then the Bode magnitude plot rolls off at high frequency. It is not hard to see that the plot of $c/(\tau s + 1)$ dominates that of $\hat{G}$ for sufficiently large positive c and sufficiently small positive τ, that is,

$$|c/(\tau j\omega + 1)| \geq |\hat{G}(j\omega)|, \qquad \forall \omega.$$

But $c/(\tau s + 1)$ has finite 2-norm; its 2-norm equals $c/\sqrt{2\tau}$ (how to do this computation is shown below). Hence $\hat{G}$ has finite 2-norm.

The rest of the proof follows similar lines. ∎

How to Compute the 2-Norm

Suppose that $\hat{G}$ is strictly proper and has no poles on the imaginary axis (so its 2-norm is finite). We have

$$\begin{aligned}
\|\hat{G}\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 d\omega \\
&= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \hat{G}(-s)\hat{G}(s)ds \\
&= \frac{1}{2\pi j}\oint \hat{G}(-s)\hat{G}(s)ds.
\end{aligned}$$

The last integral is a contour integral up the imaginary axis, then around an infinite semicircle in the left half-plane; the contribution to the integral from this semicircle equals zero because $\hat{G}$ is strictly proper. By the residue theorem, $\|\hat{G}\|_2^2$ equals the sum of the residues of $\hat{G}(-s)\hat{G}(s)$ at its poles in the left half-plane.

Example 1 Take $\hat{G}(s) = 1/(\tau s + 1)$, $\tau > 0$. The left half-plane pole of $\hat{G}(-s)\hat{G}(s)$ is at $s = -1/\tau$. The residue at this pole equals

$$\lim_{s\to -1/\tau}\left(s+\frac{1}{\tau}\right)\frac{1}{-\tau s+1}\frac{1}{\tau s+1} = \frac{1}{2\tau}.$$

Hence $\|\hat{G}\|_2 = 1/\sqrt{2\tau}$.

How to Compute the ∞-Norm

This requires a search. Set up a fine grid of frequency points,

$$\{\omega_1, \ldots, \omega_N\}.$$

Then an estimate for $\|\hat{G}\|_\infty$ is

$$\max_{1\le k\le N} |\hat{G}(j\omega_k)|.$$

Alternatively, one could find where $|\hat{G}(j\omega)|$ is maximum by solving the equation

$$\frac{d|\hat{G}|^2}{d\omega}(j\omega) = 0.$$

This derivative can be computed in closed form because $\hat{G}$ is rational. It then remains to compute the roots of a polynomial.

Example 2 Consider

$$\hat{G}(s) = \frac{as+1}{bs+1}$$

with $a, b > 0$. Look at the Bode magnitude plot: For $a \ge b$ it is increasing (high-pass); else, it is decreasing (low-pass). Thus

$$\|\hat{G}\|_\infty = \begin{cases} a/b, & a \ge b \\ 1, & a < b. \end{cases}$$

2.3 Input-Output Relationships

The question of interest in this section is: If we know how big the input is, how big is the output going to be? Consider a linear system with input u, output y, and transfer function $\hat{G}$, assumed stable and strictly proper. The results are summarized in two tables below. Suppose that u is the unit impulse, δ. Then the 2-norm of y equals the 2-norm of G, which by Parseval's theorem equals the 2-norm of $\hat{G}$; this gives entry (1,1) in Table 2.1. The rest of the first column is for the ∞-norm and *pow*, and the second column is for a sinusoidal input. The ∞ in the (1,2) entry is true as long as $\hat{G}(j\omega) \neq 0$.

	$u(t) = \delta(t)$	$u(t) = \sin(\omega t)$
$\|y\|_2$	$\|\hat{G}\|_2$	∞
$\|y\|_\infty$	$\|G\|_\infty$	$\|\hat{G}(j\omega)\|$
$pow(y)$	0	$\frac{1}{\sqrt{2}}\|\hat{G}(j\omega)\|$

Table 2.1: Output norms and *pow* for two inputs

Now suppose that u is not a fixed signal but that it can be any signal of 2-norm ≤ 1. It turns out that the least upper bound on the 2-norm of the output, that is,

$$\sup\{\|y\|_2 : \|u\|_2 \leq 1\},$$

which we can call the 2-norm/2-norm *system gain*, equals the ∞-norm of $\hat{G}$; this provides entry (1,1) in Table 2.2. The other entries are the other system gains. The ∞ in the various entries is true as long as $\hat{G} \not\equiv 0$, that is, as long as there is some ω for which $\hat{G}(j\omega) \neq 0$.

	$\|u\|_2$	$\|u\|_\infty$	$pow(u)$
$\|y\|_2$	$\|\hat{G}\|_\infty$	∞	∞
$\|y\|_\infty$	$\|\hat{G}\|_2$	$\|G\|_1$	∞
$pow(y)$	0	$\leq \|\hat{G}\|_\infty$	$\|\hat{G}\|_\infty$

Table 2.2: System Gains

A typical application of these tables is as follows. Suppose that our control analysis or design problem involves, among other things, a requirement of disturbance attenuation: The controlled system has a disturbance input, say u, whose effect on the plant output, say y, should be small. Let G denote the impulse response from u to y. The controlled system will be required to be stable, so the transfer function $\hat{G}$ will be stable. Typically, it will be strictly proper, too (or at least proper). The tables tell us how much u affects y according to various measures. For example, if u is known to be a sinusoid of fixed frequency (maybe u comes from a power source at 60 Hz), then the second column of Table 2.1 gives the relative size of y according to the three measures. More commonly, the disturbance signal will not be known *a priori*, so Table 2.2 will be more relevant.

Notice that the ∞-norm of the transfer function appears in several entries in the tables. This norm is therefore an important measure for system performance.

Example A system with transfer function $1/(10s+1)$ has a disturbance input $d(t)$ known to have the energy bound $\|d\|_2 \le 0.4$. Suppose that we want to find the best estimate of the ∞-norm of the output $y(t)$. Table 2.2 says that the 2-norm/∞-norm gain equals the 2-norm of the transfer function, which equals $1/\sqrt{20}$. Thus

$$\|y\|_\infty \le \frac{0.4}{\sqrt{20}}.$$

The next two sections concern the proofs of the tables and are therefore optional.

2.4 Power Analysis (Optional)

For a power signal u define the *autocorrelation function*

$$R_u(\tau) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t)u(t+\tau)dt,$$

that is, $R_u(\tau)$ is the average value of the product $u(t)u(t+\tau)$. Observe that

$$R_u(0) = pow(u)^2 \ge 0.$$

We must restrict our definition of a power signal to those signals for which the above limit exists for all values of τ, not just $\tau = 0$. For such signals we have the additional property that

$$|R_u(\tau)| \le R_u(0).$$

Proof The Cauchy-Schwarz inequality implies that

$$\left|\int_{-T}^{T} u(t)v(t)dt\right| \le \left(\int_{-T}^{T} u(t)^2dt\right)^{1/2} \left(\int_{-T}^{T} v(t)^2dt\right)^{1/2}.$$

Set $v(t) = u(t+\tau)$ and multiply by $1/(2T)$ to get

$$\left|\frac{1}{2T}\int_{-T}^{T} u(t)u(t+\tau)dt\right| \le \left(\frac{1}{2T}\int_{-T}^{T} u(t)^2dt\right)^{1/2} \left(\frac{1}{2T}\int_{-T}^{T} u(t+\tau)^2dt\right)^{1/2}.$$

Now let $T \to \infty$ to get the desired result. ■

Let S_u denote the Fourier transform of R_u. Thus

$$\begin{aligned}
S_u(j\omega) &= \int_{-\infty}^{\infty} R_u(\tau)\mathrm{e}^{-j\omega\tau}d\tau,\\
R_u(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_u(j\omega)\mathrm{e}^{j\omega\tau}d\omega,\\
pow(u)^2 &= R_u(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_u(j\omega)d\omega.
\end{aligned}$$

From the last equation we interpret $S_u(j\omega)/2\pi$ as power density. The function S_u is called the *power spectral density* of the signal u.

Now consider two power signals, u and v. Their *cross-correlation function* is

$$R_{uv}(\tau) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t)v(t+\tau)dt$$

and S_{uv}, the Fourier transform, is called their *cross-power spectral density function.*

We now derive some useful facts concerning a linear system with transfer function $\hat{G}$, assumed stable and proper, and its input u and output y.

1. $R_{uy} = G * R_u$

Proof Since

$$y(t) = \int_{-\infty}^{\infty} G(\alpha)u(t-\alpha)d\alpha \tag{2.1}$$

we have

$$u(t)y(t+\tau) = \int_{-\infty}^{\infty} G(\alpha)u(t)u(t+\tau-\alpha)d\alpha.$$

Thus the average value of $u(t)y(t+\tau)$ equals

$$\int_{-\infty}^{\infty} G(\alpha)R_u(\tau-\alpha)d\alpha. \blacksquare$$

2. $R_y = G * G_{\rm rev} * R_u$ where $G_{\rm rev}(t) := G(-t)$

Proof Using (2.1) we get

$$y(t)y(t+\tau) = \int_{-\infty}^{\infty} G(\alpha)y(t)u(t+\tau-\alpha)d\alpha,$$

so the average value of $y(t)y(t+\tau)$ equals

$$\int_{-\infty}^{\infty} G(\alpha)R_{yu}(\tau-\alpha)d\alpha$$

(i.e., $R_y = G * R_{yu}$). Similarly, you can check that $R_{yu} = G_{\rm rev} * R_u$. ■

3. $S_y(j\omega) = |\hat{G}(j\omega)|^2 S_u(j\omega)$

Proof From the previous fact we have

$$S_y(j\omega) = \hat{G}(j\omega)\hat{G}_{\rm rev}(j\omega)S_u(j\omega),$$

so it remains to show that the Fourier transform of $G_{\rm rev}$ equals the complex-conjugate of $\hat{G}(j\omega)$. This is easy. ■

2.5 Proofs for Tables 2.1 and 2.2 (Optional)

Table 2.1

Entry (1,1) If $u = \delta$, then $y = G$, so $\|y\|_2 = \|G\|_2$. But by Parseval's theorem, $\|G\|_2 = \|\hat{G}\|_2$.

Entry (2,1) Again, since $y = G$.

Entry (3,1)

$$\begin{aligned} pow(y)^2 &= \lim \frac{1}{2T}\int_0^T G(t)^2 dt \\ &\leq \lim \frac{1}{2T}\int_0^\infty G(t)^2 dt \\ &= \lim \frac{1}{2T}\|G\|_2^2 \\ &= 0 \end{aligned}$$

Entry (1,2) With the input $u(t) = \sin(\omega t)$, the output is

$$y(t) = |\hat{G}(j\omega)| \sin[\omega t + \arg \hat{G}(j\omega)]. \tag{2.2}$$

The 2-norm of this signal is infinite as long as $\hat{G}(j\omega) \neq 0$, that is, the system's transfer function does not have a zero at the frequency of excitation.

Entry (2,2) The amplitude of the sinusoid (2.2) equals $|\hat{G}(j\omega)|$.

Entry (3,2) Let $\phi := \arg \hat{G}(j\omega)$. Then

$$\begin{aligned} pow(y)^2 &= \lim \frac{1}{2T}\int_{-T}^{T} |\hat{G}(j\omega)|^2 \sin^2(\omega t + \phi) dt \\ &= |\hat{G}(j\omega)|^2 \lim \frac{1}{2T}\int_{-T}^{T} \sin^2(\omega t + \phi) dt \\ &= |\hat{G}(j\omega)|^2 \lim \frac{1}{2\omega T}\int_{-\omega T+\phi}^{\omega T+\phi} \sin^2(\theta) d\theta \\ &= |\hat{G}(j\omega)|^2 \frac{1}{\pi}\int_0^\pi \sin^2(\theta) d\theta \\ &= \frac{1}{2}|\hat{G}(j\omega)|^2. \end{aligned}$$

Table 2.2

Entry (1,1) First we see that $\|\hat{G}\|_\infty$ is an upper bound on the 2-norm/2-norm system gain:

$$\begin{aligned} \|y\|_2^2 &= \|\hat{y}\|_2^2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 |\hat{u}(j\omega)|^2 d\omega \\ &\leq \|\hat{G}\|_\infty^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 d\omega \\ &= \|\hat{G}\|_\infty^2 \|\hat{u}\|_2^2 \\ &= \|\hat{G}\|_\infty^2 \|u\|_2^2. \end{aligned}$$

To show that $\|\hat{G}\|_\infty$ is the least upper bound, first choose a frequency ω_o where $|\hat{G}(j\omega)|$ is maximum, that is,

$$|\hat{G}(j\omega_o)| = \|\hat{G}\|_\infty.$$

Now choose the input u so that

$$|\hat{u}(j\omega)| = \begin{cases} c, & \text{if } |\omega - \omega_o| < \epsilon \text{ or } |\omega + \omega_o| < \epsilon \\ 0, & \text{otherwise,} \end{cases}$$

where ϵ is a small positive number and c is chosen so that u has unit 2-norm (i.e., $c = \sqrt{\pi/2\epsilon}$). Then

$$\begin{aligned} \|\hat{y}\|_2^2 &\approx \frac{1}{2\pi}\left[|\hat{G}(-j\omega_o)|^2\pi + |\hat{G}(j\omega_o)|^2\pi\right] \\ &= |\hat{G}(j\omega_o)|^2 \\ &= \|\hat{G}\|_\infty^2. \end{aligned}$$

Entry (2,1) This is an application of the Cauchy-Schwarz inequality:

$$\begin{aligned} |y(t)| &= \left|\int_{-\infty}^{\infty} G(t-\tau)u(\tau)d\tau\right| \\ &\le \left(\int_{-\infty}^{\infty} G(t-\tau)^2 d\tau\right)^{1/2} \left(\int_{-\infty}^{\infty} u(\tau)^2 d\tau\right)^{1/2} \\ &= \|G\|_2\|u\|_2 \\ &= \|\hat{G}\|_2\|u\|_2. \end{aligned}$$

Hence

$$\|y\|_\infty \le \|\hat{G}\|_2\|u\|_2.$$

To show that $\|\hat{G}\|_2$ is the least upper bound, apply the input

$$u(t) = G(-t)/\|G\|_2.$$

Then $\|u\|_2 = 1$ and $|y(0)| = \|G\|_2$, so $\|y\|_\infty \ge \|G\|_2$.

Entry (3,1) If $\|u\|_2 \le 1$, then the 2-norm of y is finite [as in entry (1,1)], so $pow(y) = 0$.

Entry (1,2) Apply a sinusoidal input of unit amplitude and frequency ω such that $j\omega$ is not a zero of $\hat{G}$. Then $\|u\|_\infty = 1$, but $\|y\|_2 = \infty$.

Entry (2,2) First, $\|G\|_1$ is an upper bound on the ∞-norm/∞-norm system gain:

$$\begin{aligned} |y(t)| &= \left|\int_{-\infty}^{\infty} G(\tau)u(t-\tau)d\tau\right| \\ &\le \int_{-\infty}^{\infty} |G(\tau)u(t-\tau)|\, d\tau \\ &\le \int_{-\infty}^{\infty} |G(\tau)|\, d\tau \|u\|_\infty \\ &= \|G\|_1\|u\|_\infty. \end{aligned}$$

That $\|G\|_1$ is the least upper bound can be seen as follows. Fix t and set

$$u(t-\tau) := \operatorname{sgn}(G(\tau)), \qquad \forall \tau.$$

Then $\|u\|_\infty = 1$ and

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} G(\tau)u(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} |G(\tau)|d\tau \\ &= \|G\|_1. \end{aligned}$$

So $\|y\|_\infty \geq \|G\|_1$.

Entry (3,2) If u is a power signal and $\|u\|_\infty \leq 1$, then $pow(u) \leq 1$, so

$$\sup\{pow(y) : \|u\|_\infty \leq 1\} \leq \sup\{pow(y) : pow(u) \leq 1\}.$$

We will see in entry (3,3) that the latter supremum equals $\|\hat{G}\|_\infty$.

Entry (1,3) If u is a power signal, then from the preceding section,

$$S_y(j\omega) = |\hat{G}(j\omega)|^2 S_u(j\omega),$$

so

$$pow(y)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{G}(j\omega)|^2 S_u(j\omega)d\omega. \tag{2.3}$$

Unless $|\hat{G}(j\omega)|^2 S_u(j\omega)$ equals zero for all ω, $pow(y)$ is positive, in which case its 2-norm is infinite.

Entry (2,3) This case is not so important, so a complete proof is omitted. The main idea is this: If $pow(u) \leq 1$, then $pow(y)$ is finite but $\|y\|_\infty$ is not necessarily (see u_8 in Exercise 2). So for a proof of this entry, one should construct an input with $pow(u) \leq 1$, but such that $\|y\|_\infty = \infty$.

Entry (3,3) From (2.3) we get immediately that

$$pow(y) \leq \|\hat{G}\|_\infty pow(u).$$

To achieve equality, suppose that

$$|\hat{G}(j\omega_o)| = \|\hat{G}\|_\infty$$

and let the input be

$$u(t) = \sqrt{2}\sin(\omega_o t).$$

Then $R_u(\tau) = \cos(\omega_o \tau)$, so

$$pow(u) = R_u(0) = 1.$$

Also,

$$S_u(j\omega) = \pi\left[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)\right],$$

so from (2.3)

$$\begin{aligned} pow(y)^2 &= \frac{1}{2}|\hat{G}(j\omega_o)|^2 + \frac{1}{2}|\hat{G}(-j\omega_o)|^2 \\ &= |\hat{G}(j\omega_o)|^2 \\ &= \|\hat{G}\|_\infty^2. \end{aligned}$$

2.6 Computing by State-Space Methods (Optional)

This book is on classical control, which is set in the frequency domain. Current widespread practice, however, is to do computations using state-space methods. The purpose of this optional section is to illustrate how this is done for the problem of computing the 2-norm and ∞-norm of a transfer function. The derivation of the procedures is brief.

Consider a state-space model of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned}$$

Here $u(t)$ is the input signal and $y(t)$ the output signal, both scalar-valued. In contrast, $x(t)$ is a vector-valued function with, say, n components. The dot in $\dot{x}$ means take the derivative of each component. Then A, B, C are real matrices of sizes

$$n \times n, \quad n \times 1, \quad 1 \times n.$$

The equations are assumed to hold for $t \geq 0$. Take Laplace transforms with zero initial conditions on x:

$$\begin{aligned} s\hat{x}(s) &= A\hat{x}(s) + B\hat{u}(s), \\ \hat{y}(s) &= C\hat{x}(s). \end{aligned}$$

Now eliminate $\hat{x}(s)$ to get

$$\hat{y}(s) = C(sI - A)^{-1}B\hat{u}(s).$$

We conclude that the transfer function from $\hat{u}$ to $\hat{y}$ is

$$\hat{G}(s) = C(sI - A)^{-1}B.$$

This transfer function is strictly proper. [Try an example: start with some A, B, C with $n = 2$, and compute $\hat{G}(s)$.]

Going the other way, from a strictly proper transfer function to a state-space model, is more profound, but it is true that for every strictly proper transfer function $\hat{G}(s)$ there exist (A, B, C) such that

$$\hat{G}(s) = C(sI - A)^{-1}B.$$

From the representation

$$\hat{G}(s) = \frac{1}{\det(sI - A)} C \text{ adj}(sI - A)B$$

it should be clear that the poles of $\hat{G}(s)$ are included in the eigenvalues of A. We say that A is *stable* if all its eigenvalues lie in $\mathrm{Re}\, s < 0$, in which case $\hat{G}$ is a stable transfer function.

Now start with the representation

$$\hat{G}(s) = C(sI - A)^{-1}B$$

with A stable. We want to compute $\|\hat{G}\|_2$ and $\|\hat{G}\|_\infty$ from the data (A, B, C).

The 2-Norm

Define the matrix exponential

$$\mathrm{e}^{tA} := I + tA + \frac{t^2}{2!}A^2 + \cdots$$

just as if A were a scalar (convergence can be proved). Let a prime denote transpose and define the matrix

$$L := \int_0^\infty \mathrm{e}^{tA}BB'\mathrm{e}^{tA'}\,dt$$

(the integral converges because A is stable). Then L satisfies the equation

$$AL + LA' + BB' = 0.$$

Proof Integrate both sides of the equation

$$\frac{d}{dt}\mathrm{e}^{tA}BB'\mathrm{e}^{tA'} = A\mathrm{e}^{tA}BB'\mathrm{e}^{tA'} + \mathrm{e}^{tA}BB'\mathrm{e}^{tA'}A'$$

from 0 to ∞, noting that $\exp(tA)$ converges to 0 because A is stable, to get

$$-BB' = AL + LA'. \blacksquare$$

In terms of L a simple formula for the 2-norm of $\hat{G}$ is

$$\|\hat{G}\|_2 = (CLC')^{1/2}.$$

Proof The impulse response function is

$$G(t) = C\mathrm{e}^{tA}B, \quad t > 0.$$

Calling on Parseval we get

$$\begin{aligned}
\|\hat{G}\|_2^2 &= \|G\|_2^2 \\
&= \int_0^\infty C e^{tA} B B' e^{tA'} C' dt \\
&= C \int_0^\infty e^{tA} B B' e^{tA'} dt C' \\
&= CLC'. \blacksquare
\end{aligned}$$

So a procedure to compute the 2-norm is as follows:

Step 1 Solve the equation

$$AL + LA' + BB' = 0$$

for the matrix L.

Step 2

$$\|\hat{G}\|_2 = (CLC')^{1/2}$$

The ∞-Norm

Computing the ∞-norm is harder; we shall have to be content with a search procedure. Define the $2n \times 2n$ matrix

$$H := \begin{bmatrix} A & BB' \\ -C'C & -A' \end{bmatrix}.$$

Theorem 1 *$\|\hat{G}\|_\infty < 1$ iff H has no eigenvalues on the imaginary axis.*

Proof The proof of this theorem is a bit involved, so only sufficiency is considered, and it is only sketched.

It is not too hard to derive that

$$1/[1 - \hat{G}(-s)\hat{G}(s)] = 1 + \begin{bmatrix} 0 & B' \end{bmatrix} (sI - H)^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

Thus the poles of $[1 - \hat{G}(-s)\hat{G}(s)]^{-1}$ are contained in the eigenvalues of H.

Assume that H has no eigenvalues on the imaginary axis. Then $[1 - \hat{G}(-s)\hat{G}(s)]^{-1}$ has no poles there, so $1 - \hat{G}(-s)\hat{G}(s)$ has no zeros there, that is,

$$|\hat{G}(j\omega)| \neq 1, \quad \forall\omega.$$

Since $\hat{G}$ is strictly proper, this implies that

$$|\hat{G}(j\omega)| < 1, \quad \forall\omega$$

(i.e., $\|\hat{G}\|_\infty < 1$). $\blacksquare$

The theorem suggests this way to compute an ∞-norm: Select a positive number γ; test if $\|\hat{G}\|_\infty < \gamma$ (i.e., if $\|\gamma^{-1}\hat{G}\|_\infty < 1$) by calculating the eigenvalues of the appropriate matrix; increase or decrease γ accordingly; repeat. A bisection search is quite efficient: Get upper and lower bounds for $\|\hat{G}\|_\infty$; try γ midway between these bounds; continue.

Exercises

1. Suppose that $u(t)$ is a continuous signal whose derivative $\dot{u}(t)$ is continuous too. Which of the following qualifies as a norm for u?

$$\sup_t |\dot{u}(t)|$$

$$|u(0)| + \sup_t |\dot{u}(t)|$$

$$\max\{\sup_t |u(t)|, \sup_t |\dot{u}(t)|\}$$

$$\sup_t |u(t)| + \sup_t |\dot{u}(t)|$$

2. Consider the Venn diagram in Figure 2.1. Show that the functions u_1 to u_9, defined below, are located in the diagram as shown in Figure 2.2. All the functions are zero for $t < 0$.

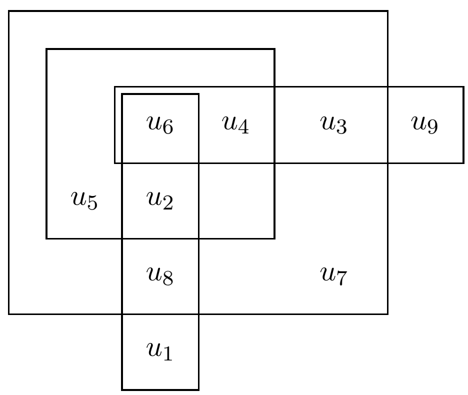

Figure 2.2: Figure for Exercise 2.

$$\begin{aligned}
u_1(t) &= \begin{cases} 1/\sqrt{t}, & \text{if } t \le 1 \\ 0, & \text{if } t > 1 \end{cases} \\
u_2(t) &= \begin{cases} 1/t^{1/4}, & \text{if } t \le 1 \\ 0, & \text{if } t > 1 \end{cases} \\
u_3(t) &= 1 \\
u_4(t) &= 1/(1+t) \\
u_5(t) &= u_2(t) + u_4(t) \\
u_6(t) &= 0 \\
u_7(t) &= u_2(t) + 1
\end{aligned}$$

For u_8, set

$$v_k(t) = \begin{cases} k, & \text{if } k < t < k + k^{-3} \\ 0, & \text{otherwise} \end{cases}$$

and then

$$u_8(t) = \sum_1^\infty v_k(t).$$

Finally, let u_9 equal 1 in the intervals

$$[2^{2k}, 2^{2k+1}], \quad k = 0, 1, 2, \ldots$$

and zero elsewhere.

3. Suppose that $\hat{G}(s)$ is a real-rational, stable transfer function with $\hat{G}^{-1}$ stable, too (i.e., neither poles nor zeros in Re $s \geq 0$). True or false: The Bode phase plot, $\angle\hat{G}(j\omega)$ versus ω, can be uniquely constructed from the Bode magnitude plot, $|\hat{G}(j\omega)|$ versus ω. (Answer: false!)

4. Recall that the transfer function for a pure timedelay of τ time units is

$$\hat{D}(s) := \mathrm{e}^{-s\tau}.$$

Say that a norm $\| \ \|$ on transfer functions is *time-delay invariant* if for every transfer function $\hat{G}$ (such that $\|\hat{G}\| < \infty$) and every $\tau > 0$,

$$\|\hat{D}\hat{G}\| = \|\hat{G}\|.$$

Is the 2-norm or ∞-norm time-delay invariant?

5. Compute the 1-norm of the impulse response corresponding to the transfer function

$$\frac{1}{\tau s + 1}, \quad \tau > 0.$$

6. For $\hat{G}$ stable and strictly proper, show that $\|G\|_1 < \infty$ and find an inequality relating $\|\hat{G}\|_\infty$ and $\|G\|_1$.

7. This concerns entry (2,2) in Table 2.2. The given entry assumes that $\hat{G}$ is stable and strictly proper. When $\hat{G}$ is stable but only proper, it can be expressed as

$$\hat{G}(s) = c + \hat{G}_1(s)$$

with c constant and $\hat{G}_1$ stable and strictly proper. Show that the correct (2,2)-entry is

$$|c| + \|G_1\|_1.$$

8. Show that entries (2,2) and (3,2) in Table 2.1 and entries (1,1), (3,2), and (3,3) in Table 2.2 hold when $\hat{G}$ is stable and proper (instead of strictly proper).

9. Let $\hat{G}(s)$ be a strictly proper stable transfer function and $G(t)$ its inverse Laplace transform. Let $u(t)$ be a signal of finite 1-norm. True or false:

$$\|G * u\|_1 \leq \|G\|_1 \|u\|_1?$$

10. Consider a system with transfer function

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta, \omega_n > 0,$$

and input

$$u(t) = \sin 0.1t, \quad -\infty < t < \infty.$$

Compute *pow* of the output.

11. Consider a system with transfer function

$$\frac{s+2}{4s+1}$$

and input u and output y. Compute

$$\sup_{\|u\|_\infty = 1} \|y\|_\infty$$

and find an input achieving this supremum.

12. For a linear system with input $u(t)$ and output $y(t)$, prove that

$$\sup_{\|u\| \le 1} \|y\| = \sup_{\|u\| = 1} \|y\|$$

where the norm is, say, the 2-norm.

13. Show that the 2-norm for transfer functions is not submultiplicative.

14. Write a MATLAB program to compute the ∞-norm of a transfer function using the grid method. Test your program on the function

$$\frac{1}{s^2 + 10^{-6} s + 1}$$

and compare your answer to the exact solution computed by hand using the derivative method.

Notes and References

The material in this chapter belongs to the field of mathematics called functional analysis. Tools from functional analysis were introduced into the subject of feedback control around 1960 by G. Zames and I. Sandberg. Some references are Desoer and Vidyasagar (1975), Holtzman (1970), Mees (1981), and Willems (1971). The state-space procedure for the ∞-norm is from Boyd et al. (1989).